Computational Algorithms and Numerical Dimensions

www.journal-cand.com

Com. Alg. Num. Dim. Vol. 4, No. 1 (2025) 65–94.

Paper Type: Original Article

Restricted and Extended Plus Operations for Soft Sets

Aslıhan Sezgin[1,*], Fitnat Nur Aybek[2], Nurcan Bilgili Güngör[3]

[1] Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye; aslihan.sezgin@amasya.edu.tr.

[2] Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye; fitnataybk.123@gmail.com.

[3] Department of Mathematics, Faculty of Arts and Science, Amasya University, Amasya, Türkiye; nurcan.bilgili@amasya.edu.tr.

Citation:

Received: 28 August 2024 *Revised: 29 October 2024* *Accepted: 15 November 2024*	Sezgin, A., Aybek, F. N., & Güngör, N. B. (2025). Restricted and extended plus operations for soft sets. *Computational algorithms and numerical dimensions*, *4*(1), 65-94.

Abstract

Soft set theory has gained importance as an innovative approach for addressing uncertainty-related problems and modeling uncertainty since it was introduced by Molodtsov in 1999. It has several applications in both theoretical and practical contexts. Soft set operations, the basic concept of the theory, have been of interest to researchers since the theory's inception, and some restricted and extended soft set operations have been defined and studied with their properties. This paper introduces a new restricted and extended soft set operation, which we refer to as restricted plus and extended plus operations. We also analyze these operations' fundamental algebraic characteristics in depth. Additionally, these operations' distributions over several soft-set operations are examined. By considering the algebraic properties of the operation and its distribution rules, we show that extended plus operation and other types of soft sets form many important algebraic structures, including semiring and near semiring in the collection of soft sets over the universe. Since the primary notion of the theory is the operations of soft sets as they serve not only the decision-making processes as accurately as possible but also the foundation for various applications, including cryptology, this theoretical study is of great importance both theoretically and practically.

Keywords: Soft sets, Soft set operations, Restricted plus operation, Extended plus operation.

1 | Introduction

There are many uncertainties in the real world, which vary from person to person. For example, expressions like freezing, low speed, beautiful baby, and short height, which we use daily, differ from person to person. Many other events in our lives contain similar uncertainties. And classical mathematical logic is not sufficient to meet these uncertainties. Therefore, some scientific studies beyond the known methods have been needed to eliminate these uncertainties. In this regard, in the early 17^{th} century, probability theory first examined the

Corresponding Author: aslihan.sezgin@amasya.edu.tr

https://doi.org/10.22105/cand.2024.475779.1107

uncertainty situation mathematically by putting forward the probability theory. In the early 19th century, many scientists conducted studies on uncertainty. In 1920 and 1930s, the concept of uncertainty was explained by opening the door to multiple values and three-valued logic system, respectively. Theories that can be used to explain uncertainty include fuzzy set theory, interval mathematics, and probability theory; however, each of these theories has drawbacks. Therefore, in 1999, Molodtsov [1] proposed the theory of Soft Set (SS), which is independent of the construction of the membership function. Unlike fuzzy set theory, SS theory does not have a real-valued function but rather a set-valued function, aiming to eliminate uncertainty. This theory has been successfully applied to several mathematical disciplines since its inception. Measurement theory, game theory, probability theory, Riemann integration, and Perron integration process study are some disciplines.

The first studies on SS operations were carried out by Maji et al. [2] as well as Pei and Miao [3]. Ali et al. [4] developed a number of SS operations, including restricted and extended SS operations. In their work on SSs, Sezgin and Yavuz [5] defined and gave characteristics of the restricted symmetric difference of SSs. Additionally, they looked through the fundamentals of SS operations and showed the relationships between each other. The algebraic structures of SSs were thoroughly examined by Ali et al. [6]. A number of researchers [7–16] were interested in SS operations and studied the subject in depth. Over the last five years, many new types of SS operations have been proposed. Eren and Çalışıcı [17] introduced the definition and examined the properties of the soft binary piecewise difference operation in SSs. Sezgin and Çağman [18] introduced the extended difference of SSs, while Stojanović [19] gave the definition and looked into the properties of the extended symmetric difference of SSs, Sezgin and Sarıalioğlu [20] carried out a complete examination of restricted and extended symmetric difference operations, too.

Sezgin et al. [21] worked on some new binary set operations. They described several additional ones by being motivated by the study of Çağman [22], who added two new complement operations to the literature. Aybek [23] defined many additional restricted and extended SS operations with the help of the operations defined in [21], [22]. In [24–26] new complementary extended SS operations were defined.

In the scope of algebra, one of the most essential mathematical issues is analyzing the properties of the operation defined on a set to classify algebraic structures. In this regard, proposing new SS operations, examining their properties, and thinking about which algebraic structures they form in the collection of SSs help us comprehend the theory better. Up to now, there have been defined four restricted SS operations as restricted intersection, union, difference, and symmetric difference and four extended SS operations as extended intersection, union, difference, and symmetric difference for SSs.

In this study, by presenting new restricted and extended SS operations as "restricted plus operation and extended plus operations of SSs" and carefully analyzing the algebraic structures connected to it as well as other SS operations within the collection of SSs, we hope to advance the subject of SS theory significantly. This study is organized as follows. Section 2 recalls the basic concepts of SSs and several algebraic structures. In Section 3, the new SS operations are defined. First of all, restricted plus SS operation and then extended plus SS operation are introduced, and the algebraic properties of these new SS operations are thoroughly examined. However, we explore the distribution laws of these operations over other types of SS operations, such as restricted, extended, and soft binary piecewise operations. Considering the distribution laws and the algebraic properties of the SS operations, a detailed analysis of the algebraic structures generated by the set of SSs with these operations is provided.

We demonstrate how several significant algebraic structures, including semiring and near semiring, are formed in the collection of SSs over the universe using restricted and extended SS operations since such a thorough examination improves our understanding of the implications and uses of SS theory in a variety of fields. In the conclusion section, we discuss the significance of the study results and their potential applications to the subject.

2 | Preliminaries

Several algebraic structures and several fundamental ideas in SS theory are provided in this section.

Definition 1. Let U be the universal set, E be the parameter set, P(U) be the power set of U, and T⊆ E. A pair (¥, T) is called a SS on U. Here, ¥ is a function given by $¥ : T \rightarrow P(U)$ [27].

Throughout this paper, the collection of all the SSs over U (no matter what the parameter set is) is designated by S_E(U) and S_T(U) denotes the collection of all SSs over U with a fixed parameter set T, where T is a subset of E.

Definition 2. Let (¥,T) be a SS over U. If ¥(x)=∅ for all x∈T, then the SS (¥,T) is called a null SS with respect to K, denoted by $\emptyset_K$. Similarly, let (¥,E) be a SS over U. If ¥(x)=∅ for all x∈E, then the SS (¥,E) is called a null SS with respect to E, denoted by $\emptyset_E$ [4]. A SS with an empty parameter set is denoted as $\emptyset_\emptyset$. It is obvious that $\emptyset_\emptyset$ is the only SS with an empty parameter set [6].

Definition 3. Let (¥,T) be a SS over U. If ¥(x)=U for all x∈T, then the SS (¥,T) is called a relative whole SS with respect to T, denoted by U_T. Similarly, let (¥,E) be a SS over U. If ¥(x)=U for all x∈E, then the SS (¥,E) is called an absolute SS, and denoted by U_E [4].

Definition 4. Let (¥,T) and (₲,Y) be SSs over U. If T⊆Y and for all x∈T, ¥(x) ⊆₲(x), then (¥,T) is said to be a soft subset of (₲,Y), denoted by (¥,T)$\widetilde{\subseteq}$(₲,Y). If (₲,Y) is a soft subset of (¥,T), then (¥,T) is said to be a soft superset of (₲,Y), denoted by (¥,T)$\widetilde{\supseteq}$(₲,Y). If (¥,T)$\widetilde{\subseteq}$(₲,Y) and (₲,Y)$\widetilde{\subseteq}$(¥,T), then (¥,T) and (₲,Y) are called soft equal sets [3].

Definition 5. Let (¥,T) be a SS over U. The relative complement of (¥,T), denoted by $(¥, T)^r$=($¥^r$,T), is defined as follows: $¥^r$(x)=U-¥(x) for all x∈T [4].

Çağman [22] introduced two new complements: the inclusive and exclusive complements, which we denote as + and θ, respectively. For two sets X and Y, these binary operations are defined as X+Y=X'∪Y and XθY=X'∩Y'. Sezgin et al. [21] investigated the relationship between these two operations and also introduced three new binary operations: for two sets X and Y, these new operations are defined as X*Y=X'∪Y', XγY= X'∩Y, X**λ**Y=X∪Y' [21]. Let "⋈" be used to represent the set operations (i.e., there ⋈ can be ∩, ∪,\, Δ, +,θ, *, λ,γ), then all types of SS operations are defined as follows:

Definition 6. Let (¥, T) and (₲, Y) be two SSs over U. The restricted ⋈ operation of (¥, T) and (₲, Y) is the (ℋ, Z), denoted by (¥, T) $\bowtie_R$ (₲, Y)= (ℋ, Z), where Z = T ∩ Y≠ ∅ and for all x∈ Z, ℋ(x) = ¥(x) ⋈₲(x). Here, if Z= T ∩ Y = ∅, then (¥, T) $\bowtie_R$(₲, Y)= $\emptyset_\emptyset$ [4], [23].

Definition 7. Let (¥, T) and (₲, Y) be two SSs over U. The extended ⋈ operation (¥, T) and (₲,Y) is the SS (ℋ,Ƶ), denoted by (¥, T) $\bowtie_\varepsilon$(₲, Y) = (ℋ, Z), where Z= T ∪ Y, and for all x∈ Z,

$$\mathfrak{H}(x) = \begin{cases} F(x). & x \in T - Y. \\ G(x). & x \in Y - T. \\ F(x) \bowtie G(x). & x \in T \cap Y, \end{cases} \tag{1}$$

[2], [4], [19], [23].

Definition 8. Let (¥,T) and (₲, Y) be two SSs over U. The complementary extended $\bowtie_\varepsilon$ operation (¥,T) and (₲,Y) is the SS (ℋ,Ƶ), denoted by (¥, T) $\overset{*}{\bowtie}_\varepsilon$(₲, Y) = (ℋ, Z), where Z = T ∪ Y, and for all x∈ Z,

$$\mathfrak{H}(x) = \begin{cases} F'^{(x)}. & x \in T - Y. \\ G'^{(x)}. & x \in Y - T. \\ F(x) \bowtie G(x). & x \in T \cap Y, \end{cases}$$

[24–26]

Definition 9. Let (¥,T) and (₲,Y) be two SSs on U. The soft binary piecewise ⋈ operation of (¥,T) and (₲,Y) is the SS (ℋ,T), denoted by $(¥, T)\underset{\bowtie}{\sim}(₲, Y) = (\mathfrak{H}, T)$, where for all x∈ T,

$$\mathfrak{H}(x) = \begin{cases} F(x). & x \in T - Y. \\ ¥(x) \bowtie G(x). & x \in T \cap Y, \end{cases}$$

[5], [17], [28].

Definition 10. Let (¥,T) and (₲,Y) be two SSs on U. The complementary soft binary piecewise ⋈ operation of (¥,T) and (₲,Y) is the SS (ℋ,T), denoted by $(¥, T) \underset{\bowtie}{\overset{*}{\sim}} (₲, Y) = (\mathfrak{H}, T)$, where for all x∈ T,

$$\mathfrak{H}(x) = \begin{cases} F'^{(x)}. & x \in T - Y. \\ ¥(x) \bowtie G(x). & x \in T \cap Y, \end{cases}$$

[18], [20]. For more about SSs, we refer to [29-54].

Definition 11. An element s ∈S is called idempotent if s2=s. If s2=s for all s∈S, then the algebraic structure (S,⋆) is said to be idempotent. An idempotent semigroup is called a band, an idempotent and commutative semigroup is called a semilattice, and an idempotent and commutative monoid is called a bounded semilattice [55].

Definition 12. Let S be a non-empty set, and let "+" and "⋆" be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a semiring:

I. S, +) is a semigroup.

II. (S, ⋆) is a semigroup.

III. For all x, y, z ∈S, x⋆(y + z) = x⋆y + x⋆z and (x +y) ⋆z = x⋆z + y⋆z.

If x+y=y+z for all x,y∈S, then S is called an additive commutative semiring. If x⋆y=y⋆x for all x,y∈S, then S is called a multiplicative commutative semiring. If there exists an element 1∈S such that x⋆1=1⋆x=x for all x∈S (multiplicative identity), then S is called semiring with unity. If there exists 0∈S such that 0⋆x=x⋆0=0 and 0+x=x+0=x for all x∈S, then 0 is called the zero of S. A semiring with a commutative addition and a zero element is called a hemiring [56].

Definition 13. Let S be a non-empty set, and let "+" and "⋆" be two binary operations defined on S. If the algebraic structure (S, +, ⋆) satisfies the following properties, then it is called a near semiring (or seminearring):

I. (S,+) is a semigroup.

II. (S, ⋆) is a semigroup.

III. For all x,y,z ∈ S, (x+y) ⋆z = x⋆z+y∗z (right distributivity).

If the additive zero element 0 of S (that is, for all x∈S, 0+x=0+x=x) satisfies that for all x∈S, 0⋆x=0 (left absorbing element), then (S, +, ⋆) is called a (right) near semiring with zero. If (S, +, ⋆) additionally satisfies x⋆0=0 for all x∈S (right absorbing element), then it is called a zero symmetric near semiring [57]. A near semiring is a more general algebraic structure than semiring. For more about possible applications of graphs and network research concerning SSs, we refer to [58].

3 | Restrıcted and Extended Plus Operatıon

In this section, new restricted and extended SS operations called the restricted plus and extended plus operation, are introduced, and we explore the distributive rules of these operations over other types of SSs. Thus, the relationships of these operations with other SS operations, along with their algebraic properties, are discussed. It also investigated which algebraic structures these operations form in collection $S_E(U)$, leading to meaningful results.

3.1 | Restricted Plus Operation and Its Properties

Definition 14. Let (¥, T) and (₲, ₴) be SSs over U. The restricted plus of (¥, T) and (₲, ₴), denoted by (F, T)$+_R$(G, Z), is defined as (F, T)$+_R$(G, Z) = (H, C), where C=T∩₴, and if C=T∩₴≠∅, then for all ƛ∈C, ϑϹ(ƛ)= ¥(ƛ) +₲(ƛ)=¥'(ƛ)∪₲(ƛ); if C= T∩₴=∅, then (¥,T) $+_R$ (₲,₴)=(ϑϹ, C)= $\emptyset_\emptyset$.

Since the only SS with an empty parameter set is $\emptyset_\emptyset$, if C=T∩₴= ∅, then it is obvious that (¥,T) $+_R$ (₲,₴)= $\emptyset_\emptyset$. Thus, in order to define the restricted plus operation of (¥, T) and (₲, ₴), there is no extra condition as T∩₴≠ ∅.

Example 1. Let E={e_1,e_2,e_3,e_4} be the parameter set, T={e_1,e_3} and Z={e_2,e_3,e_4} be subsets of E, U={h_1,h_2, h_3,h_4, h_5} be the universal set, (¥,T) and (₲,₴) be the SSs over U as (¥,T)={(e_1,{h_2,h_5),(e_3,{h_1,h_2,h_5})},(₲,₴)={(e_2,{h_1,h_4,h_5}),{(e_3,{h_2,h_3,h_4}),(e_4, h_3,h_5})}. Here let (¥,T)$+_R$(₲,₴)=(ϑϹ,T∩ ₴), where for all ƛ ∈ T ∩ ₴={e_3}. Thus, ϑϹ(ƛ)=F′(ƛ) ∪₲(ƛ), ϑϹ(e_3)=¥'(e_3)∪₲(e_3)={h_3,h_4}∪{h_2,h_3,h_4}={h_2, h_3, h_4}.Thus,

(¥, T)$+_R$(₲, ₴) = {(e_3, {h_2, h_3, h_4})},

Theorem 1. (Algebraic properties of the operation)

I. The set S_E(U) in closed under $+_R$.

Proof: It is clear that $+_R$ is a binary operation in SE(U). That is,

$+_R$: S_E(U)x S_E(U) → S_E(U)
((¥, T), (₲, ₴)) → (¥, T)$+_R$(₲, ₴) = (ℌ, T ∩ ₴),

Similarly,

$+_R$: S_T(U) x S_T(U) → S_T(U)
((¥, T), (₲, T)) → (¥, T)$+_R$(₲, T) = (ℌ, T ∩ T) = (ℌ, T),

That is, let T be a fixed subset of the set E and (¥,T) and (₲,T) be elements of ST(U). Then so is (¥,T)$+_R$ (₲,T). Namely, S_T(U) is closed under $+_R$ as well.

II. Let (¥,T), (₲,₴) and (ϑϹ,Ȣ) be SSs over U. Then, ([(¥,T)$+_R$ (₲,₴)] $+_R$ (ϑϹ,Ȣ) ≠ (¥,T) $+_R$ [(₲,₴) $+_R$ (ϑϹ,Ȣ)].

Proof: Let (¥,T)$+_R$(₲,₴)=(S,T∩₴), where for all ƛ∈T∩₴, T(ƛ)=¥'(ƛ)∪₲(ƛ). Let (S,T∩₴)

$+_R$(ℌ, Ȣ) =(R,(T∩₴)∩Ȣ)), where for all ƛ∈(T∩₴)∩Ȣ, R(ƛ)=T'(ƛ)∪ϑϹ(ƛ). Thus,

R(ƛ) = [¥(ƛ) ∩ ₲'(ƛ)] ∪ ℌ(ƛ),

Let (₲,₴) $+_R$(ϑϹ,Ȣ)=(K,₴∩Ȣ), where for all ƛ∈₴∩Ȣ, K(ƛ)=₲'(ƛ)∪ϑϹ(ƛ). Let (¥,T) $+_R$ (K, ₴ ∩Ȣ) =(S,T∩(₴∩Ȣ)), where for all ƛ∈T∩(₴∩Ȣ), S(ƛ)=¥'(ƛ)∪K(ƛ). Thus,

S(ƛ) = ¥'(ƛ) ∪ [₲'(ƛ) ∪ ℌ(ƛ)],

Thus, (R,(T ∩ ₴) ∩ Ȣ) ≠(S, T∩(₴∩Ȣ)). That is, in S_E(U), the operation $+_R$ is not associative. Here, it is obvious that if T∩₴= ∅ or ₴∩Ȣ = ∅ or T∩ Ȣ = ∅, then as both sides of the equality are $\emptyset_\emptyset$, the operation $+_R$ is associative under these conditions.

III. Let (¥,T), (₲,T) and (ϑϹ,T) be SSs over U. Then, [(¥,T) $+_R$(₲,T)] $+_R$(ϑϹ,T)≠(¥,T)$+_R$[(₲,T)$+_R$(ϑϹ,T)].

Proof: Let (¥,T)$+_R$(₲,T)=(K,T), where for all ƛ∈T∩T=T, K(ƛ)=¥'(ƛ)∪₲(ƛ). Let (K,T)$+_R$

(ℌ,T) =(R,T), where for all ƛ∈T∩T=T, R(ƛ)=K'(ƛ)∪ϑϹ(ƛ). Hence,

R(ƛ) = [F (ƛ) ∩ ₲'(ƛ)] ∪ ℌ(ƛ),

Let (Ɠ,T) $+_R$(ℋ,T)=(L,T), where for all ƛ∈T∩T, L(ƛ)=Ɠ'(ƛ)∪ℋ(ƛ). Let (¥,T) $+_R$ (L,T)= (N,T), where for all ƛ∈T∩T, N(ƛ)= ¥'(ƛ)∪L(ƛ). Hence,

$$\mathrm{N}(ƛ) = ¥'(ƛ) \cup [Ɠ'(ƛ) \cup ℌ(ƛ)].$$

thus, (R,T)≠(N,T). That is, $+_R$ is not associative in the collection of SSs with a fixed parameter set.

IV. Let (¥,T) and (Ɠ,ƻ) be SSs over U. Then, (¥,T)$+_R$(Ɠ,ƻ)≠(Ɠ,ƻ) $+_R$(¥,T) where where T∩ƻ≠ ∅.

Proof: Let (¥,T) $+_R$(Ɠ,ƻ)=(ℋ,T∩ƻ), where for all ƛ∈T∩ƻ, ℋ(ƛ)=¥'(ƛ)∪Ɠ(ƛ). Let (Ɠ,ƻ)$+_R$ (¥,T)=(S,ƻ∩T), where for all ƛ∈ƻ∩T, S(ƛ)=Ɠ'(ƛ)∪¥(ƛ). Thus,

$$(¥, \mathrm{T}) +_R (Ɠ, ƻ) \neq (Ɠ, ƻ) +_R (¥, \mathrm{T}).$$

that is, $+_R$ is not commutative in S_E(U). Here, it is evident that if T∩ƻ= ∅, then since both sides are $\emptyset_\emptyset$, $+_R$ is commutative in S_E(U) under this condition. Moreover, it is evident that (¥,T)$+_R$(Ɠ,T)≠(Ɠ,T) $+_R$(¥,T), namely, $+_R$ is not commutative in the collection of SSs with a fixed parameter set.

V. Let (¥,T) be a SS over U. Then, (¥,T) $+_R$ (¥,T)= U_T.

Proof: Let (¥,T) $+_R$(¥,T)=(ℋ,T∩T). Thus, for all ƛ∈T, ℋ(ƛ)=¥'(ƛ)∪¥(ƛ)=U. Hence (ℋ,T)= U_T.

That is, the operation $+_R$ is not idempotent in S_E(U).

VI. Let (¥,T) be a SS over U. Then, (¥,T) $+_R$ $\emptyset_T$=$(¥,\mathrm{T})^r$.

Proof: Let $\emptyset_T$=(S,T), where for all ƛ∈T, S(ƛ)= ∅. Let (¥,T) $+_R$(S,T)=(ℋ,T∩T), where for all ƛ∈T, ℋ(ƛ)=¥'(ƛ)∪S(ƛ)=¥'(ƛ)∪∅=¥'(ƛ). Thus, (ℋ,T)= $(¥,\mathrm{T})^r$.

VII. Let (¥,T) be a SS over U. Then, $\emptyset_T$$+_R$(¥,T)= U_T.

Proof: Let $\emptyset_T$=(S,T), where for all ƛ∈T, S(ƛ)=∅. Let (S,T) $+_R$ (¥,T)=(ℋ,T∩T), where for all ƛ∈T, ℋ(ƛ)=S'(ƛ)∪¥(ƛ)=U∪¥(ƛ)=U. Thus, (ℋ,T)= U_T.

VIII. Let (¥,T) be a SS over U. Then, (¥,T) $+_R$ $\emptyset_ℵ$=$(¥,\mathrm{T} \cap ℵ)^r$.

Proof: Let $\emptyset_M$=(S,ℵ), where for all ƛ∈ℵ, S(ƛ)=∅. Let (¥,T)$+_R$(S,ℵ)=(ℋ,T∩ℵ), where for all ƛ∈T∩ℵ, ℋ(ƛ)=¥'(ƛ)∪S(ƛ)=¥'(ƛ)∪∅=¥'(ƛ). Thus, (ℋ,T∩ℵ)= $(¥,\mathrm{T} \cap ℵ)^r$.

IX. Let (¥,T) be a SS over U. Then, $\emptyset_ℵ$ $+_R$ (¥,T)=$U_{ℵ\cap T}$.

Proof: Let $\emptyset_ℵ$=(S,ℵ), where for all ƛ∈ℵ, S(ƛ)=∅. Let (S,ℵ) $+_R$ (¥,T)=(ℋ,ℵ∩T), where for all ƛ∈ℵ∩T, ℋ(ƛ)=S'(ƛ)∪¥(ƛ)=U∪ ¥(ƛ)=U. Thus, (ℋ,ℵ∩T)= $U_{ℵ\cap T}$.

X. Let (¥,T) be a SS over U. Then, (¥,T)$+_R$ $\emptyset_E$=$(¥,\mathrm{T})^r$.

Proof: Let $\emptyset_E$=(S,E), where for all ƛ∈E, S(ƛ)=∅. Let (¥,T) $+_R$ (S,E)=(ℋ, T∩E), where for all ƛ∈T∩E=T, ℋ(ƛ)=¥'(ƛ)∪S(ƛ)=¥'(ƛ)∪∅= ¥'(ƛ). Thus, (ℋ,T)= $(¥,\mathrm{T})^r$.

XI. Let (¥,T) be a SS over U. Then, $\emptyset_E$ $+_R$ (¥,T)= U_T.

Proof: Let $\emptyset_E$ =(S,E), where for all ƛ∈E, S(ƛ)=∅. Let (S,E) $+_R$(¥,T)=(ℋ,E∩T), where for all ƛ∈E∩T=T, ℋ(ƛ)=S'(ƛ)∪¥(ƛ)=U∪¥(ƛ)=U. Thus, (ℋ,T)= U_T.

XII. Let (¥,T) be a SS over U. Then, (¥,T) $+_R$ $\emptyset_\emptyset$=$\emptyset_\emptyset$$+_R$ (¥,T)= $\emptyset_\emptyset$.

Proof: Let $\emptyset_\emptyset$ =(S,∅). Thus, (¥,T) $+_R$(S,∅)=(ℋ,T ∩ ∅)=(ℋ, ∅). Since $\emptyset_\emptyset$ is the only SS with the empty parameter set, (ℋ, ∅)=$\emptyset_\emptyset$. That is, the absorbing element of $+_R$ in S_E(U) is the SS $\emptyset_\emptyset$.

XIII. Let (¥,T) be a SS over U. Then, (¥,T) $+_R$$U_T$= U_T.

Proof: Let U_T=(K,T), where for all ƛ∈T, K(ƛ)=U. Let (¥,T) $+_R$(K,T)=(ℋ,T∩T), where for all ƛ∈T, ℋ(ƛ)= ¥'(ƛ)∪T(ƛ)=¥'(ƛ)∪U=U. Thus, (ℋ,T)= U_T. That is, the right absorbing element of $+_R$ in S_T(U) is the SS U_T.

XIV. Let (¥,T) be a SS over U. Then, $U_T$$+_R$ (¥,T)= (¥,T).

Proof: Let U_T=(K,T), where for all ƛ∈T, K(ƛ)=U. Let (K,T) $+_R$(¥,T)=(ℋ,T∩T), where for all ƛ∈T, ℋ(ƛ)=T'(ƛ)∪¥(ƛ)=∅∪¥(ƛ)=¥(ƛ). Thus, (ℋ,T)=(¥,T). That is, the left identity element of $+_R$ in S_T(U) is the SS U_T.

XV. Let (¥,T) be a SS over U. Then, (¥,T)$+_R$$U_{Ꝩ}$=$U_{T∩Ꝩ}$.

Proof: Let $U_{Ꝩ}$=(K,Ꝩ). Thus, for all ƛ∈Ꝩ, K(ƛ)=U. Let (¥,T) $+_R$ (K,Ꝩ)=(ℋ,T∩Ꝩ), where for all ƛ∈T∩Ꝩ, ℋ(ƛ)=¥'(ƛ)∪T(ƛ)=¥'(ƛ)∪U=U. Thus, (ℋ, T∩Ꝩ) =$U_{T∩Ꝩ}$.

XVI. Let (¥,T) be a SS over U. Then, $U_{Ꝩ}$ $+_R$(¥,T)=(¥,T∩Ꝩ).

Proof: Let $U_{Ꝩ}$=(K,Ꝩ), where for all ƛ∈Ꝩ, K(ƛ)=U. Let (K,Ꝩ)$+_R$(¥,T)=(ℋ,Ꝩ∩T), where for all ƛ∈Ꝩ∩T, ℋ(ƛ)=T'(ƛ)∪¥(ƛ)=∅∪¥(ƛ)= ¥(ƛ). Thus, (ℋ, Ꝩ∩T)=(¥, Ꝩ∩T).

XVII. Let (¥,T) be a SS over U. Then, (¥,T) $+_R$ U_E=U_T.

Proof: Let U_E=(K,E), where for all ƛ∈E, K(ƛ)=U. Let (¥,T) $+_R$(K,E)=(ℋ, T∩E), where for all ƛ∈T∩E=T,ℋ(ƛ)=¥'(ƛ)∪K(ƛ)=¥'(ƛ)∪U=U.Hence (ℋ,T)= U_T.

XVIII. Let (¥,T) be a SS over U. Then, U_E $+_R$ (¥,T)= (¥,T).

Proof: Let U_E=(K,E), where for all ƛ∈E, K(ƛ)=U. Let (K,E)$+_R$(¥,T)=(ℋ, E∩T), where for all ƛ∈E∩T=T, ℋ(ƛ)=T'(ƛ)∪¥(ƛ)=∅∪¥(ƛ)= ¥(ƛ). Thus, (ℋ,T)=(¥,T). That is, the left identity element of $+_R$ in S_E(U) is the SS U_E.

XIX. Let (¥,T) be a SS over U. Then, (¥,T)$+_R$ (¥,T)r=(¥,T)r.

Proof: Let (¥,T)r=(ℋ,T), where for all ƛ∈T, ℋ(ƛ)=¥'(ƛ). Let (¥,T)$+_R$(ℌ,T)=(L,T∩T), where for all ƛ∈T, L(ƛ)=¥'(ƛ)∪ℋ(ƛ)=¥'(ƛ)∪¥'(ƛ)=¥'(ƛ). Thus, (L,T)=(¥,T)r. That is, every relative complement of the SS is its own right absorbing element for $+_R$ in S_E(U).

XX. Let (¥,T) be a SS over U. Then, (¥,T)r $+_R$(¥,T)=(¥,T).

Proof: Let (¥,T)r=(ℋ,T), where for all ƛ∈T, ℋ(ƛ)=¥'(ƛ). Let (ℋ,T) $+_R$(¥,T)=(L,T∩T), where for all ƛ∈T, T(ƛ)=ℋ'(ƛ)∪¥(ƛ)=¥(ƛ)∪¥(ƛ)=¥(ƛ). Thus, (L,T)=(¥,T). That is, every relative complement of the SS is its own left identity element for $+_R$ in S_E(U).

XXI. Let (¥,T) and (Ǥ,Ƶ) be SSs over U. Then, [(¥,T) $+_R$(Ǥ,Ƶ)]r=(¥,T) $\backslash_R$ (Ǥ,Ƶ).

Proof: Let (¥,T)$+_R$(Ǥ,Ƶ)=(ℋ,T∩Ƶ), where for all ƛ∈T∩Ƶ, ℋ(ƛ)=¥'(ƛ)∪Ǥ(ƛ). Let (ℋ,T∩Ƶ)r=(K,T∩Ƶ), where for all ƛ∈T∩Ƶ, K(ƛ)=¥(ƛ)∩Ǥ'(ƛ). Thus, (K, T∩Ƶ)=(¥,T) $\backslash_R$ (Ǥ,Ƶ). Here, if T∩Ƶ=∅, then both sides is the SS $∅_∅$, and so the equality is again satisfied.

XXII. Let (¥,T) and (Ǥ,T) be SSs over U. Then, (¥,T) $+_R$(Ǥ,T)= $∅_T$ ⇔ (¥,T) =U_T and (G,T) = $∅_T$.

Proof: Let (¥,T) $+_R$(Ǥ,T) = (K,T∩T), where for all ƛ∈T, K(ƛ)= ¥'(ƛ)∪Ǥ(ƛ). Since (K,T)= $∅_T$, for all ƛ∈T, K(ƛ)= ∅. Thus, for all ƛ∈T, K(ƛ)=¥'(ƛ)∪Ǥ(ƛ)=∅⇔ for all ƛ ∈ T, ¥'(ƛ)= ∅ and Ǥ(ƛ)= ∅ ⇔ for all ƛ∈T, ¥(ƛ)=U ve Ǥ(ƛ)= ∅ ⇔ (¥,T) = U_T and (Ǥ,T)= $∅_T$.

XXIII. Let (¥,T) and (Ǥ,Ƶ) be SSs over U. Then, $∅_{T∩Ƶ}$ ⊆̃(¥,T)$+_R$(Ǥ,Ƶ), (¥,T) $+_R$ (Ǥ,Ƶ) ⊆̃ U_T and (¥,T) $+_R$ (Ǥ,Ƶ) ⊆̃ U_Z, where T∩Ƶ≠ ∅.

XXIV. Let (¥,T) and (Ǥ,Ƶ) be SSs over U. Then) (¥,T∩Ƶ)r ⊆̃ (¥,T) $+_R$(Ǥ,Ƶ) and (Ǥ,T∩Ƶ) ⊆̃ (¥,T) $+_R$(Ǥ,Ƶ), where T∩Ƶ≠ ∅.

Proof: Let (¥,T) $+_R$ (₢,ƻ)=(ℋ,T∩ƻ), where for all ƛ∈T∩ƻ, ℋ(ƛ)=F'∪₢(ƛ). Since, for all ƛ∈T∩ƻ, ¥'(ƛ)⊆ ¥'(ƛ) ∪ ₢(ƛ)=ℋ(ƛ) and ₢(ƛ)⊆ ¥'(ƛ) ∪ ₢(ƛ)=ℋ(ƛ). Thus, (¥,T∩ƻ)r⊆̃ (¥,T) $+_R$(₢,ƻ) and (₢,T∩ƻ) ⊆̃ (¥,T) $+_R$(₢,ƻ).

XXV. Let (¥,T) and (₢,T) be SSs over U. Then, (¥,T)r ⊆̃ (¥,T) $+_R$(₢,T) ve (₢,T) ⊆̃(¥,T) $+_R$(₢,T).

Proof: Let (¥,T) $+_R$(₢,T)=(ℋ,T∩T), where for all ƛ∈T, ℋ(ƛ)=¥'(ƛ)∪₢(ƛ). Since for all ƛ∈T, ¥'(ƛ)⊆¥'(ƛ)∪₢(ƛ)=ℋ(ƛ), thus (¥,T)r ⊆̃ (¥,T) $+_R$(₢,T). Similarly, since ₢(ƛ)⊆ ¥'(ƛ)∪₢(ƛ)=ℋ(ƛ), thus (₢,T) ⊆̃(¥,T) $+_R$ (₢,T).

XXVI. Let (¥,T), (₢,T) and (ℋ,ƻ) be SSs over U. If (¥,T) ⊆̃ (₢,T), then (₢,T) $+_R$(ℋ,ƻ) ⊆̃(¥,T) $+_R$(ℋ,ƻ) and (ℋ,ƻ) $+_R$(¥,T) ⊆̃(ℋ,ƻ) $+_R$(₢,T).

Proof: Let (¥,T)⊆̃ (₢,T). Then for all ƛ∈T, ¥(ƛ)⊆ ₢(ƛ), thus for all ƛ∈T, ₢'(ƛ)⊆ F'(ƛ). Let (₢,T) $+_R$(ℋ,ƻ)=(K,T∩ƻ), where for all ƛ∈T∩ƻ, K(ƛ)=₢'(ƛ)∪ℋ(ƛ). Let (¥,T) $+_R$(ℋ,ƻ)=(L,T∩ƻ), where for all ƛ∈T∩ƻ, L(ƛ)=¥'(ƛ)∪ℋ(ƛ). Since for all ƛ∈T∩ƻ, K(ƛ)=₢'(ƛ)∪ℋ(ƛ)⊆¥'(ƛ)∪ℋ(ƛ)=L(ƛ), (₢,T)$+_R$(ℋ,ƻ)⊆̃(¥,T) $+_R$(ℋ,ƻ). Also, since for all ƛ∈ ƻ∩T, ℋ'(ƛ)∪¥(ƛ)⊆ℋ'(ƛ)∪₢(ƛ), (ℋ,ƻ) $+_R$(¥,T) ⊆̃(ℋ,ƻ) $+_R$(₢,T). Here, if T∩ƻ=∅, then both sides is the SS $\emptyset_\emptyset$, and so the property is again satisfied.

XXVII. Let (¥,T), (₢,T) and (ℋ,ƻ) be SSs over U. If (₢,T) $+_R$(ℋ,ƻ) ⊆̃(¥,T) $+_R$(ℋ,ƻ), then (¥,T) ⊆̃ (₢,T) needs not be true. That is, the converse of *Theorem 1* (XXVI) is not true. Similarly, if (ℋ,ƻ)$+_R$(¥,T) ⊆̃(ℋ,ƻ) $+_R$(₢,T), (¥,T) ⊆̃ (₢,T) needs not be true.

Proof: We give a counterexample to show that the converse of *Theorem 1* (XXVI) is not true. Let E={e_1,e_2,e_3,e_4, e_5} be the parameter set, T={e_1,e_3} and Z={e_1,e_3,e_5} be the subsets of E, U={h_1,h_2, h_3,h_4, h_5} be the universal set and (¥,T), (₢,T) and (ℋ,ƻ) be the SSs as follows:

(¥,T)={(e_1,{h_2, h_5,}),(e_3,{h_1,h_2,h_5,})},(₢,T)={(e_1,{h_2}),(e_3,{h_1,h_2})},(ℋ,ƻ)={(e_1,U),(e_3,U),(e_5,{h_2})}.

Let (₢,T) $+_R$(ℋ,ƻ)=(L,T∩ ƻ), where for all ƛ ∈ T ∩ ƻ={e_1, e_3}, L(ƛ)=G′(ƛ) ∪ℋ(ƛ), L(e_1)=₢'(e_1)∪ℋ(e_1)=U, L(e_3)=₢'(e_3)∪ℋ(e_3)=U. Thus, (₢,T)$+_R$(ℋ,ƻ)={(e_1,U), (e_3,U)}.

Now let (¥,T)$+_R$(ℋ,ƻ)=(K,T∩ ƻ), where for all ƛ ∈ T ∩ Z={e_1, e_3}, K(ƛ)=F′(ƛ) ∪ℋ(ƛ), K(e_1)=¥'(e_1)∪ℋ(e_1) =U, K(e_3)=¥'(e_3)∪ℋ(e_3)=U. Thus, (¥,T) $+_R$(ℋ,ƻ)={(e_1,U), (e_3,U)}.

It is observed that (₢,T) $+_R$(ℋ,ƻ) ⊆̃(¥,T) $+_R$(ℋ,ƻ); however, (¥,T) is not a soft subset of (₢,T). Similarly, one can show that if (ℋ,ƻ) $+_R$(¥,T) ⊆̃(ℋ,ƻ) $+_R$(₢,T), then (¥,T) ⊆̃ (₢,T) needs not to be true.

XXVIII. Let (¥,T), (₢,T), (K,V) and (L,V) be SSs over U. Then If (¥,T)⊆̃ (₢,T) and (K,V)⊆̃ (L,V), (₢,T) $+_R$(K,V) ⊆̃(¥,T) $+_R$(L,V) and (L,V) $+_R$(¥,T) ⊆̃(K,V) $+_R$(₢,T).

Proof: Let (¥,T) ⊆̃ (G,T) and (K,V) ⊆̃ (L,V). Thus, for all ƛ∈T and for all ƛ∈ƻ, ¥(ƛ)⊆ G(α) and K(ƛ)⊆ L(α). Hence, for all ƛ∈T, ₢' (ƛ)⊆ F′(α) and for all ƛ∈ƻ, L'(ƛ)⊆ K′(ƛ). Let (₢,T) $+_R$(K,V)=(M,T∩ V). Thus, for all ƛ∈T∩ V, M(ƛ)=₢'(ƛ)∪K(ƛ). Let (¥,T) $+_R$(L,V)=(N,T∩ V). Thus, for all ƛ∈T∩ V, N(ƛ)=¥'(ƛ)∪L(ƛ). Since, for all ƛ∈T∩ V, ₢'(ƛ)⊆ F′(ƛ) and K(ƛ)⊆ L(α), M(ƛ)=₢'(ƛ)∪K(ƛ)⊆¥'(ƛ)∪L(ƛ)=N(ƛ). Thus, (₢,T) $+_R$(K,V) ⊆̃(¥,T) $+_R$(L,V). Under similar conditions, since for all ƛ∈V∩ T, L' (ƛ)∪¥(ƛ)⊆K' (ƛ)∪₢(ƛ), (L,V) $+_R$(¥,T) ⊆̃(K,V) $+_R$(₢,T) can be illustrated similarly. Here, if T∩V=∅, then both sides are the SS $\emptyset_\emptyset$, and so the property is again satisfied.

Theorem 2. Let (¥,T), (₢,ƻ), and (ℋ,Ȣ) be SSs over U. Then, restricted plus operation distributes over other restricted SS operations as follows:

LHS distributions

I. (¥,T) $+_R$ [(₢,ƻ)$\cap_R$(ℋ,Ȣ)] = [(¥,T) $+_R$(₢,ƻ)] $\cap_R$ [(¥,T) $+_R$(ℋ,Ȣ)].

Proof: consider first the LHS. Let (Ⓖ,Ƶ)$\cap_R$(ℋ,Ⴘ)=(R,Ƶ∩Ⴘ), where for all λ∈Ƶ∩Ⴘ, R(λ)=Ⓖ(λ)∩ℋ(λ). Let (¥,T) $+_R$ (R,Ƶ∩Ⴘ)=(N,T∩(Ƶ∩Ⴘ)), where for all λ∈T∩(Ƶ∩Ⴘ), N(λ)= ¥'(λ)∪R(λ). Thus, for all λ∈T∩Ƶ∩Ⴘ,

N(λ) = ¥'(λ) ∪ [(Ⓖ(λ) ∩ ℌ(λ)].

Now consider the RHS, i.e. [(¥,T)$+_R$ (Ⓖ,Ƶ)] $\cap_R$ [(¥,T)$+_R$ (ℋ,Ⴘ)]. Let (¥,T) $+_R$ (Ⓖ,Ƶ)=(V,T∩Ƶ), where for all λ∈T∩Ƶ, V(λ)=¥'(λ)∪Ⓖ(λ) and let (¥,T)$+_R$ (ℋ,Ⴘ)=(W,T∩Ⴘ), where for all λ∈T∩Ⴘ, W(λ)=¥'(λ)∪ℋ(λ). Let (V,T∩Ƶ)$\cap_R$(W,T∩Ⴘ)=(S,(T∩Ƶ)∩(T∩Ⴘ)), where for all λ∈T∩Ƶ∩Ⴘ, S(λ)=V(λ)∩W(λ). Thus,

S(λ) = [¥'(λ) ∪ Ⓖ(λ)] ∩ [¥'(λ) ∪ ℌ(λ)],

Hence, (N,T∩Ƶ∩Ⴘ)=(S,T∩Ƶ∩Ⴘ). Here, if T∩Ƶ=∅ or T∩Ⴘ=∅ or Ƶ∩Ⴘ=∅, then both sides is $\emptyset_\emptyset$. Thus, the equality is satisfied in all circumstances.

II. (¥,T) $+_R$[(Ⓖ,Ƶ) $\cup_R$ (ℋ,Ⴘ)] = [(¥,T)$+_R$(Ⓖ,Ƶ)] $\cup_R$ [(¥,T) $+_R$ (ℋ,Ⴘ)].

III. (¥,T) $+_R$[(Ⓖ,Ƶ) $*_R$ (ℋ,Ⴘ)] = [(¥,T) $*_R$(Ⓖ,Ƶ)] $\cup_R$ [(¥,T) $*_R$ (ℋ,Ⴘ)].

IV. (¥,T) $+_R$[(Ⓖ,Ƶ) θ_R (ℋ,Ⴘ)] = [(¥,T) $*_R$(Ⓖ,Ƶ)] $\cap_R$ [(¥,T) $*_R$ (ℋ,Ⴘ)].

RHS distributions

I. [(¥,T) $\cup_R$ (Ⓖ,Ƶ)] $+_R$(ℋ,Ⴘ)=[(¥,T)$+_R$(ℋ,Ⴘ)] $\cap_R$ [(Ⓖ,Ƶ) $+_R$ (ℋ,Ⴘ)].

Proof: consider first the LHS. Let (¥,T) $\cup_R$(Ⓖ,Ƶ)=(R,T∩Ƶ), where for all λ∈T∩Ƶ, R(λ)=¥(λ)∪Ⓖ(λ). Let (R,T∩Ƶ) $+_R$(ℋ,Ⴘ)=(N,(T∩Ƶ)∩Ⴘ)), where for all λ∈(T∩Ƶ)∩Ⴘ, N(λ)=R'(λ)∪ℋ(λ). Thus,

N(λ) = [¥'(λ) ∩ Ⓖ'(λ)] ∪ ℌ(λ),

Now consider the RHS, i.e. [(¥,T)$+_R$(ℋ,Ⴘ)] $\cap_R$ [(Ⓖ,Ƶ) $+_R$ (ℋ,Ⴘ)]. Let (¥,T)$+_R$(ℋ,Ⴘ)=(S,T∩Ⴘ), where for all λ∈T∩Ⴘ, T(λ)=¥'(λ)∪ℋ(λ) and let (Ⓖ,Ƶ)$+_R$(ℋ,Ⴘ)= (K,Ƶ∩Ⴘ), where for all λ∈Ƶ∩Ⴘ, K(λ)=Ⓖ'(λ)∪ℋ(λ). Assume that (S,T∩Ƶ) $\cap_R$(K,Ƶ∩Ⴘ)=(L,(T∩Ƶ∩Ⴘ)), where for all λ∈(T∩Ƶ)∩(Ƶ∩Ⴘ), L(λ)=S(λ)∩K(λ). Thus,

L(λ) = [¥'(λ) ∪ ℌ(λ)] ∩ [Ⓖ'(λ) ∪ ℌ(λ)],

Hence, (N,T∩Ƶ∩Ⴘ)=(L,T∩Ƶ∩Ⴘ). Here, if T∩Ƶ=∅ or T∩Ⴘ=∅ or Ƶ∩Ⴘ=∅, then both sides is $\emptyset_\emptyset$. Thus, the equality is satisfied in all circumstances.

II. [(¥,T) $\cap_R$ (Ⓖ,Ƶ)] $+_R$ (ℋ,Ⴘ)=[(¥,T) $+_R$(ℋ,Ⴘ)] $\cup_R$[(Ⓖ,Ƶ) $+_R$(ℋ,Ⴘ)].

III. [(¥,T) θ_R (Ⓖ,Ƶ)] $+_R$ (ℋ,Ⴘ)= [(¥,T)$\cup_R$(ℋ,Ⴘ)] $\cup_R$ [(Ⓖ,Ƶ) $\cup_R$ (ℋ,Ⴘ)].

IV. [(¥,T) $*_R$(Ⓖ,Ƶ)] $+_R$(ℋ,Ⴘ)= [(¥,T) $\cup_R$ (ℋ,Ⴘ)] $\cap_R$ [(Ⓖ,Ƶ) $\cup_R$ (ℋ,Ⴘ)].

Theorem 3. Let (¥,T), (Ⓖ,Ƶ), and (ℋ,Ⴘ) be SSs over U. Then, restricted plus operation distributes over extended SS operations as follows:

LHS distributions

I. (¥,T) $+_R$[(Ⓖ,Ƶ) $\cap_\varepsilon$ (ℋ,Ⴘ)] = [(¥,T)$+_R$(Ⓖ,Ƶ)] $\cap_\varepsilon$ [(¥,T) $+_R$ (ℋ,Ⴘ)].

Proof: consider first the LHS. Let (Ⓖ,Ƶ) $\cap_\varepsilon$ (ℋ,Ⴘ)=(R,Ƶ∪Ⴘ), where for all λ∈Ƶ∪Ⴘ,

$$R(\lambda)=\begin{cases} Ⓖ(\lambda), & \lambda\in Ƶ-Ⴘ,\\ ℋ(\lambda), & \lambda\in Ⴘ-Ƶ,\\ Ⓖ(\lambda)\cap ℋ(\lambda), & \lambda\in Ƶ\cap Ⴘ.\end{cases}$$

Let (¥,T)$+_R$(R,Ƶ∪Ⴘ) =(N,(T∩(Ƶ∪Ⴘ)), where for all λ∈T∩(Ƶ∪Ⴘ), N(λ)=¥'(λ)∪R(λ). Thus,

$$\text{N(ƛ)}=\begin{cases}\text{¥'(ƛ)∪Ǥ(ƛ),} & \text{ƛ∈T∩(ƻ-Ꝩ)=T∩ƻ∩Ꝩ',}\\ \text{¥'(ƛ)∪ℋ(ƛ),} & \text{ƛ∈T∩(Ꝩ-ƻ)=T∩ƻ'∩Ꝩ,}\\ \text{¥'(ƛ)∪[Ǥ(ƛ)∩ℋ(ƛ)],} & \text{ƛ∈T∩(ƻ∩Ꝩ)=T∩ƻ∩Ꝩ.}\end{cases}$$

Now consider the RHS, i.e. [(¥,T)$+_R$(Ǥ,ƻ)] $\cap_\varepsilon$[(¥,T) $+_R$ (ℋ,Ꝩ)]. Let (¥,T)$+_R$(Ǥ,ƻ)=(K,T∩ƻ), where for all ƛ∈T∩ƻ, K(ƛ)=¥'(ƛ)∪Ǥ(ƛ) and let (¥,T)$+_R$(ℋ,Ꝩ)=(S,T∩Ꝩ), where for all ƛ∈T∩Ꝩ, S(ƛ)= ¥'(ƛ)∪ℋ(ƛ). Let (K,T∩ƻ) $\cap_\varepsilon$(S,T∩Ꝩ)=(L,(T∩ƻ)∪(T∩Ꝩ)), where for all ƛ∈(T∩ƻ)∪(T∩Ꝩ),

$$\text{L(ƛ)}=\begin{cases}\text{K(ƛ),} & \text{ƛ∈(T∩ƻ)-(T∩Ꝩ)=T∩(ƻ-Ꝩ),}\\ \text{S(ƛ),} & \text{ƛ∈(T∩Ꝩ)-(T∩ƻ)=T∩(Ꝩ-ƻ),}\\ \text{K(ƛ)∩S(ƛ),} & \text{ƛ∈(T∩ƻ)∩(T∩Ꝩ)=T∩(ƻ∩Ꝩ).}\end{cases}$$

Thus,

$$\text{L(ƛ)}=\begin{cases}\text{¥'(ƛ)∪Ǥ(ƛ),} & \text{ƛ∈T∩ƻ∩Ꝩ',}\\ \text{¥'(ƛ)∪ℋ(ƛ),} & \text{ƛ∈T∩ƻ'∩Ꝩ,}\\ \text{[¥'(ƛ)∪Ǥ(ƛ)]∩[¥'(ƛ)∪ℋ(ƛ)],} & \text{ƛ∈T∩ƻ∩Ꝩ.}\end{cases}$$

Hence, (N, T∩(ƻ∪Ꝩ))=(L, (T∩ƻ)∪(T∩Ꝩ)). Here, if T∩ƻ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪ℋ(ƛ), and if T∩Ꝩ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪Ǥ(ƛ). Thus, there is no extra condition as T∩ƻ≠ ∅ and/or T∩Ꝩ=≠ ∅ for satisfying *Theorem 3* (I).

II. (¥,T) $+_R$[(Ǥ,ƻ) $\cup_\varepsilon$ (ℋ,Ꝩ)] = [(¥,T)$+_R$(Ǥ,ƻ)] $\cup_\varepsilon$[(¥,T) $+_R$ (ℋ,Ꝩ)].

RHS distributions

I. [(¥,T)$\cup_\varepsilon$(Ǥ,ƻ)]$+_R$(ℋ,Ꝩ)= [(¥,T)$+_R$(ℋ,Ꝩ)] $\cap_\varepsilon$ [(Ǥ,ƻ) $+_R$ (ℋ,Ꝩ)].

Proof: consider first the LHS. Let (¥,T) $\cup_\varepsilon$(Ǥ,ƻ)=(R,T∪ƻ), where for all ƛ∈ T∪ƻ,

$$\text{R(ƛ)} =\begin{cases}\text{¥(ƛ),} & \text{ƛ∈T-ƻ,}\\ \text{Ǥ(ƛ),} & \text{ƛ∈ƻ-T,}\\ \text{¥(ƛ)∪Ǥ(ƛ),} & \text{ƛ∈T.}\end{cases}$$

Assume that (R,T∪ƻ)$+_R$(ℋ,Ꝩ)=(N,(T∪ƻ)∩Ꝩ)), where for all ƛ∈(T∪ƻ)∩Ꝩ, N(ƛ)=R'(ƛ)∪ℋ(ƛ). Thus,

$$\text{N(ƛ)}=\begin{cases}\text{¥'(ƛ)∪ℋ(ƛ),} & \text{ƛ∈(T-ƻ)∩Ꝩ=T∩ƻ'∩Ꝩ,}\\ \text{Ǥ'(ƛ)∪ℋ(ƛ),} & \text{ƛ∈(ƻ-T)∩Ꝩ=T'∩ƻ∩Ꝩ,}\\ \text{[¥'(ƛ)∩Ǥ'(ƛ)]∪ℋ(ƛ),} & \text{ƛ∈(T∩ƻ)∩Ꝩ=T∩ƻ∩Ꝩ.}\end{cases}$$

Now consider the RHS, i.e. [(¥,T)$+_R$(ℋ,Ꝩ)]$\cap_\varepsilon$[(Ǥ,ƻ) $+_R$ (ℋ,Ꝩ)]. Let (¥,T)$+_R$ (ℋ,Ꝩ)=(K,T∩Ꝩ), where for all ƛ∈T∩Ꝩ, K(ƛ)=¥'(ƛ)∪ℋ(ƛ) and let (Ǥ,ƻ) $+_R$ (ℋ,Ꝩ)=(S,ƻ∩Ꝩ), where for all ƛ∈ƻ∩Ꝩ, S(ƛ)= Ǥ'(ƛ)∪ℋ(ƛ). Let (K,T∩Ꝩ) $\cap_\varepsilon$(S,ƻ∩Ꝩ)= (L,((T∩Ꝩ)∪(ƻ∩Ꝩ)). Hence,

$$\text{L(ƛ)}=\begin{cases}\text{K(ƛ),} & \text{ƛ∈(T∩Ꝩ)-(ƻ∩Ꝩ)=(T-ƻ)∩Ꝩ,}\\ \text{S(ƛ),} & \text{ƛ∈(ƻ∩Ꝩ)-(T∩Ꝩ)=(ƻ-T)∩Ꝩ,}\\ \text{K(ƛ)∩S(ƛ),} & \text{ƛ∈(T∩Ꝩ)∩(ƻ∩Ꝩ)=(T∩ƻ)∩Ꝩ.}\end{cases}$$

Thus,

$$\text{L(ƛ)}=\begin{cases}\text{¥'(ƛ)∪ℋ(ƛ),} & \text{ƛ∈T∩ƻ'∩Ꝩ,}\\ \text{Ǥ'(ƛ)∪ℋ(ƛ),} & \text{ƛ∈T'∩ƻ∩Ꝩ,}\\ \text{[¥'(ƛ)∪ℋ(ƛ)]∩[Ǥ'(ƛ)∪ℋ(ƛ)],} & \text{ƛ∈T∩ƻ∩Ꝩ.}\end{cases}$$

Therefore, (N,(T∪ƻ)∩Ꝩ))=(L,(T∩Ꝩ)∪(ƻ∩Ꝩ)). Here, if T∩ƻ=∅ and ƛ∈T∩ƻ'∩Ꝩ, then N(ƛ)=L(ƛ)=¥'(ƛ)∪ℋ(ƛ) and if T∩ƻ=∅ and ƛ∈T'∩ƻ∩Ꝩ, the N(ƛ)=L(ƛ)=Ꞡ'(ƛ)∪ℋ(ƛ). Furthermore, if ƻ∩Ꝩ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪ℋ(ƛ). Thus, there is no extra condition as T∩ƻ≠ ∅ and/or ƻ∩Ꝩ ≠ ∅ for satisfying *Theorem 3* (II).

II. [(¥,T) $\cap_\varepsilon$(Ꞡ,ƻ)]$+_R$(ℋ,Ꝩ)= [(¥,T)$+_R$(ℋ,Ꝩ)] $\cup_\varepsilon$ [(Ꞡ,ƻ) $+_R$ (ℋ,Ꝩ)].

Theorem 4. Let (¥,T), (Ꞡ,ƻ), and (ℋ,Ꝩ) be SSs over U. Then, restricted plus operation distributes over complimentary extended SS operations as follows:

LHS distributions

I. (¥,T) $+_R$[(Ꞡ,ƻ) $\overset{*}{*}_\varepsilon$ (ℋ,Ꝩ)] = [(¥,T) $*_R$(Ꞡ,ƻ)] $\cup_\varepsilon$[(¥,T) $*_R$ (ℋ,Ꝩ)].

Proof: consider first the LHS. Let (Ꞡ,ƻ) $\overset{*}{*}_\varepsilon$(ℋ,Ꝩ)=(R,ƻ∪Ꝩ), where for all ƛ∈ƻ∪Ꝩ,

$$R(ƛ)=\begin{cases} Ꞡ'(ƛ), & ƛ\in ƻ-Ꝩ,\\ ℋ'(ƛ), & ƛ\in Ꝩ-ƻ,\\ Ꞡ'(ƛ)\cup ℋ'(ƛ), & ƛ\in ƻ\cap Ꝩ.\end{cases}$$

Let (¥,T) $+_R$(R,ƻ∪Ꝩ) =(N,(T∩(ƻ∪Ꝩ)), where for all ƛ∈T∩(ƻ∪Ꝩ), N(ƛ)=¥'(ƛ)∪R(ƛ). Thus,

$$N(ƛ)=\begin{cases} ¥'(ƛ)\cup Ꞡ'(ƛ). & ƛ\in T\cap(ƻ-Ꝩ)=T\cap ƻ\cap Ꝩ'.\\ ¥'(ƛ)\cup ℌ'(ƛ). & ƛ\in T\cap(Ꝩ-ƻ)=T\cap ƻ'\cap Ꝩ.\\ Ꞡ'(ƛ)\cup ℌ¥'(ƛ)\cup\ [Ꞡ'(ƛ)\cup ℌ'(ƛ)]. & ƛ\in T\cap(ƻ\cap Ꝩ)=T\cap ƻ\cap Ꝩ'(ƛ).\quad ƛ\in ƻ\cap Ꝩ.\end{cases}$$

Now consider the RHS, i.e. (¥,T) $*_R$(Ꞡ,ƻ)=(K,T∩ƻ), where for all ƛ∈T∩ƻ, K(ƛ)=¥'(ƛ)∪Ꞡ'(ƛ). Let (¥,T) $*_R$(ℋ,Ꝩ)=(S,T∩Ꝩ), where for all ƛ∈T∩Ꝩ, S(ƛ)=¥'(ƛ)∪ℋ'(ƛ). Assume that (K,T∩ƻ)$\cup_\varepsilon$(S,T∩Ꝩ)=(L,(T∩ƻ)∪(T∩Ꝩ)), where for all ƛ∈(T∩ƻ)∪(T∩Ꝩ),

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ\in(T\cap ƻ)-(T\cap Ꝩ)=T\cap(ƻ-Ꝩ),\\ S(ƛ), & ƛ\in(T\cap Ꝩ)-(T\cap ƻ)=T\cap(Ꝩ-ƻ),\\ K(ƛ)\cup S(ƛ), & ƛ\in(T\cap ƻ)\cap(T\cap Ꝩ)=T\cap(ƻ\cap Ꝩ).\end{cases}$$

Thus,

$$L(ƛ)=\begin{cases} ¥'(ƛ)\cup Ꞡ'(ƛ), & ƛ\in T\cap ƻ\cap Ꝩ',\\ ¥'(ƛ)\cup ℋ'(ƛ), & ƛ\in T\cap ƻ'\cap Ꝩ,\\ [¥'(ƛ)\cup Ꞡ'(ƛ)]\cup[¥'(ƛ)\cup ℋ'(ƛ)], & ƛ\in T\cap ƻ\cap Ꝩ.\end{cases}$$

Therefore, (N,(T∩(ƻ∪Ꝩ))=(L,(T∩ƻ)∪(T∩Ꝩ)).

Here, if T∩ƻ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪ℋ'(ƛ), and if T∩Ꝩ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪Ꞡ'(ƛ). Thus, there is no extra condition as T∩ƻ≠ ∅ and/or T∩Ꝩ=≠ ∅ for satisfying *Theorem 3* (I).

II. (¥,T) $+_R$ [(Ꞡ,ƻ) $\overset{*}{\theta}_\varepsilon$ (ℋ,Ꝩ)] = [(¥,T) $*_R$(Ꞡ,ƻ)] $\cap_R$ [(¥,T) $*_R$ (ℋ,Ꝩ)].

RHS distributions

I. [(¥,T) $\overset{*}{\theta}_\varepsilon$(Ꞡ,ƻ)] $+_R$(ℋ,Ꝩ)= [(¥,T)$\cup_R$(Ꞡ,ƻ)] $\cup_\varepsilon$[(Ꞡ,ƻ) $\cup_R$ (ℋ,Ꝩ)].

Proof: consider first the LHS. Let (¥,T) $\overset{*}{\theta}_\varepsilon$ (Ꞡ,ƻ)=(R,T∪ƻ), where for all ƛ∈T∪ƻ,

$$R(ƛ)=\begin{cases} ¥'(ƛ), & ƛ\in T-ƻ,\\ Ꞡ'(ƛ), & ƛ\in ƻ-T,\\ ¥'(ƛ)\cap Ꞡ'(ƛ). & \end{cases}$$

Let (R,T∪Ƨ)$+_R$(ϑ𝓒,Ɣ) =(N,(T∪Ƨ)∩Ɣ), where for all ƛ∈(T∪Ƨ)∩Ɣ, N(ƛ)=R'(ƛ)∪ϑ𝓒(ƛ). Thus,

$$N(ƛ)=\begin{cases} ¥(ƛ)∪ϑ𝓒(ƛ), & ƛ∈(T\text{-}Ƨ)∩Ɣ=T∩Ƨ'∩Ɣ, \\ Ꞓ(ƛ)∪ϑ𝓒(ƛ), & ƛ∈(Ƨ\text{-}T)∩Ɣ=T'∩Ƨ∩Ɣ, \\ [¥(ƛ)∪Ꞓ(ƛ)]∪ϑ𝓒(ƛ), & ƛ∈(T∩Ƨ)∩Ɣ=T∩Ƨ∩Ɣ. \end{cases}$$

Now consider the RHS, i.e. [(¥,T)$∪_R$(Ꞓ,Ƨ)]$∪_ε$ [(Ꞓ,Ƨ) $∪_R$ (ϑ𝓒,Ɣ)]. Let (¥,T)$∪_R$(ϑ𝓒,Ɣ)=(K,T∩Ɣ), where for all ƛ∈T∩Ɣ, K(ƛ)=¥(ƛ)∪ϑ𝓒(ƛ) and let (Ꞓ,Ƨ) $∪_R$(ϑ𝓒,Ɣ)=(S,Ƨ∩Ɣ), where for all ƛ∈Ƨ∩Ɣ, S(ƛ)=Ꞓ(ƛ)∪ϑ𝓒(ƛ). Assume that (K,T∩Ɣ)$∪_ε$(S,Ƨ∩Ɣ)=(L,(T∩Ɣ)∪(Ƨ∩Ɣ)), where for all ƛ∈(T∩Ɣ)∪(Ƨ∩Ɣ),

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ∈(T∩Ɣ)\text{-}(Ƨ∩Ɣ)=(T\text{-}Ƨ)∩Ɣ, \\ S(ƛ), & ƛ∈(Ƨ∩Ɣ)\text{-}(T∩Ɣ)=(Ƨ\text{-}T)∩Ɣ, \\ K(ƛ)∪S(ƛ), & ƛ∈(T∩Ɣ)∩(Ƨ∩Ɣ)=(T∩Ƨ)∩Ɣ. \end{cases}$$

Thus,

$$L(ƛ)=\begin{cases} ¥(ƛ)∪ϑ𝓒(ƛ), & ƛ∈T∩Ƨ'∩Ɣ, \\ Ꞓ(ƛ)∪ϑ𝓒(ƛ), & ƛ∈T'∩Ƨ∩Ɣ, \\ [¥(ƛ)∪ϑ𝓒(ƛ)]∪[Ꞓ(ƛ)∪ϑ𝓒(ƛ)], & ƛ∈T∩Ƨ∩Ɣ. \end{cases}$$

Therefore, (N,(T∪Ƨ)∩Ɣ)=(L,(T∩Ɣ)∪(Ƨ∩Ɣ)). Here, if T∩Ƨ=∅ and ƛ∈T∩Ƨ'∩Ɣ, then N(ƛ)=L(ƛ)=¥(ƛ)∪ϑ𝓒(ƛ) and if T∩Ƨ=∅ and ƛ∈T'∩Ƨ∩Ɣ, the N(ƛ)=L(ƛ)=Ꞓ(ƛ)∪ϑ𝓒(ƛ). Furthermore, if Ƨ∩Ɣ=∅, then N(ƛ)=L(ƛ)=¥(ƛ)∪ϑ𝓒(ƛ). Thus, there is no extra condition as T∩Ƨ≠ ∅ and/or Ƨ∩Ɣ ≠ ∅ for satisfying the condition.

II. [(¥,T) $\overset{*}{*}_ε$ (Ꞓ,Ƨ)]$+_R$(ϑ𝓒,Ɣ)= [(¥,T)$∪_R$(Ꞓ,Ƨ)] $∩_ε$ [(Ꞓ,Ƨ) $∪_R$ (ϑ𝓒,Ɣ)].

Theorem 5. Let (¥,T), (Ꞓ,Ƨ), and (ϑ𝓒,Ɣ) be SSs over U. Then, restricted plus operation distributes over soft binary piecewise operations as follows:

LHS distributions

I. (¥,T) $+_R$[(Ꞓ,Ƨ) $\tilde{∩}$ (ϑ𝓒,Ɣ)]=[(¥,T)$+_R$(Ꞓ,Ƨ)] $\tilde{∩}$ [(¥,T) $+_R$ (ϑ𝓒,Ɣ)].

Proof: consider first the LHS. Let (Ꞓ,Ƨ$\tilde{∩}$) (ϑ𝓒,Ɣ)=(R,Ƨ), where for all ƛ∈Ƨ,

$$R(ƛ)=\begin{cases} Ꞓ(ƛ), & ƛ∈Ƨ\text{-}Ɣ, \\ Ꞓ(ƛ)∩ϑ𝓒(ƛ), & ƛ∈Ƨ∩Ɣ. \end{cases}$$

Let (¥,T) $+_R$ (R,Ƨ) =(N,T∩Ƨ), where for all ƛ∈T∩Ƨ, N(ƛ)=¥'(ƛ)∪R(ƛ). Thus,

$$N(ƛ)=\begin{cases} ¥'(ƛ)∪Ꞓ(ƛ), & ƛ∈T∩(Ƨ\text{-}Ɣ)=T∩Ƨ∩Ɣ', \\ ¥'(ƛ)∪[Ꞓ(ƛ)∩ϑ𝓒(ƛ)], & ƛ∈T∩(Ƨ∩Ɣ)=T∩Ƨ∩Ɣ. \end{cases}$$

Now consider the RHS, i.e., [(¥,T)$+_R$(Ꞓ,Ƨ)] $\tilde{∩}$ [(¥,T) $+_R$ (ϑ𝓒,Ɣ)].(¥,T)$+_R$(Ꞓ, Ƨ)=(K,T∩Ƨ), where for all ƛ∈T∩Ƨ, K(ƛ)=¥'(ƛ)∪Ꞓ(ƛ). Let (¥,T) $+_R$ (ϑ𝓒,Ɣ)=(S,T∩Ɣ), where for all ƛ∈T∩Ɣ, S(ƛ)=¥'(ƛ)∪ϑ𝓒(ƛ) and assume that (K,T∩Ƨ)$\tilde{∩}$ (S,T∩Ɣ)=(L,T∩Ƨ), where for all ƛ∈T∩Ƨ,

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ∈(T∩Ƨ)\text{-}(T∩Ɣ)=T∩(Ƨ\text{-}Ɣ), \\ K(ƛ)∩S(ƛ), & ƛ∈(T∩Ƨ)∩(T∩Ɣ)=T∩(Ƨ∩Ɣ). \end{cases}$$

Thus,

$$
\mathrm{L}(ƛ)=\begin{cases} ¥'(ƛ)∪Ǥ(ƛ), & ƛ∈T∩Ƶ∩ℵ', \\ [¥'(ƛ)∪Ǥ(ƛ)]∩[¥'(ƛ)∪ℋ(ƛ)], & ƛ∈T∩Ƶ∩ℵ. \end{cases}
$$

Hence (N,T∩Ƶ) =(L,T∩Ƶ). Here, if T∩Ƶ=∅, then (N,T∩Ƶ)=(L,T∩Ƶ)=$∅_∅$, and if T∩ℵ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪Ǥ(ƛ). Thus, there is no extra condition as T∩Ƶ≠ ∅ and/or T∩ ℵ ≠ ∅ for satisfying *Theorem 5* (I).

II. (¥,T) $+_R$[(Ǥ,Ƶ) $\tilde{∪}$ (ℋ,ℵ)]=[(¥,T)$+_R$(Ǥ,Ƶ)] $\tilde{∪}$ [(¥,T) $+_R$ (ℋ,ℵ)].

RHS distributions

I. [(¥,T) $\tilde{∪}$ (Ǥ,Ƶ)]$+_R$(ℋ,ℵ)= [(¥,T)$+_R$(ℋ,ℵ)] $\tilde{∩}$ [(Ǥ,Ƶ) $+_R$ (ℋ,ℵ)].

Proof: consider first the LHS. Let (¥,T) $\tilde{∪}$ (Ǥ,Ƶ)=(R,T), where for all ƛ∈T,

$$
\mathrm{R}(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T\text{-}Ƶ, \\ ¥(ƛ)∪Ǥ(ƛ), & ƛ∈T∩Ƶ. \end{cases}
$$

Let (R,T) $+_R$(ℋ,ℵ) =(N,T∩ℵ), where for all ƛ∈T∩ℵ, N(ƛ)=R'(ƛ)∪ℋ(ƛ). Thus,

$$
\mathrm{N}(ƛ)=\begin{cases} ¥'(ƛ)∪ℋ(ƛ), & ƛ∈(T\text{-}Ƶ)∩ℵ=T∩Ƶ'∩ℵ, \\ [¥'(ƛ)∩Ǥ'(ƛ)]∪ℋ(ƛ), & ƛ∈(T∩Ƶ)∩ℵ=T∩Ƶ∩ℵ. \end{cases}
$$

Now consider the RHS. Let (¥,T)$+_R$(ℋ,ℵ)=(K,T∩ℵ), where for all ƛ∈T∩ℵ, K(ƛ)=¥'(ƛ)∪ℋ(ƛ). Assume that (Ǥ,Ƶ) $+_R$ (ℋ,ℵ)=(S,Ƶ∩ℵ), where for all ƛ∈Ƶ∩ℵ, S(ƛ)=Ǥ'(ƛ)∪ℋ(ƛ) and let (K,T∩ℵ) $\tilde{∩}$(S,Ƶ∩ℵ)=(L,T∩ℵ), where for all ƛ∈T∩ℵ,

$$
\mathrm{L}(ƛ)=\begin{cases} K(ƛ), & ƛ∈(T∩ℵ)\text{-}(Ƶ∩ℵ)=(T\text{-}Ƶ)∩ℵ, \\ K(ƛ)∩S(ƛ), & ƛ∈(T∩ℵ)∩(Ƶ∩ℵ)=(T∩Ƶ)∩ℵ. \end{cases}
$$

Hence,

$$
\mathrm{L}(ƛ)=\begin{cases} ¥'(ƛ)∪ℋ(ƛ), & ƛ∈T∩Ƶ'∩ℵ, \\ [¥'(ƛ)∪ℋ(ƛ)]∩ [Ǥ'(ƛ)∪ℋ(ƛ)], & ƛ∈T∩Ƶ∩ℵ. \end{cases}
$$

Thus, (N,T∩ℵ)=(L,T∩ℵ). Here, if T∩ℵ=∅, then (N,T∩ℵ)=(L,T∩ℵ)=$∅_∅$, and if Ƶ∩ℵ=∅, then N(ƛ)=L(ƛ)=¥'(ƛ)∪ℋ(ƛ). Thus, there is no extra condition as T∩ ℵ ≠ ∅ and/or Ƶ∩ℵ ≠ ∅ for satisfying *Theorem 5* (II).

II. [(¥,T) $\tilde{∩}$ (Ǥ,Ƶ)]$+_R$(ℋ,ℵ)= [(¥,T)$+_R$(ℋ,ℵ)] $\tilde{∪}$ [(Ǥ,Ƶ) $+_R$ (ℋ,ℵ)].

3.2 | Extended Plus Operation and Its Properties

Definition 15. Let (F, T) and (G, Z) be SSs over U. The extended plus operation of (¥,T) and (G, Z) is the SS (ℋ,C), denoted by (¥,T)$+_ε$(Ǥ,Ƶ)=(ℋ,C), where C=T∪Ƶ and for all ƛ∈C,

$$
ℋ(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T\text{-}Ƶ, \\ Ǥ(ƛ), & ƛ∈Ƶ\text{-}T, \\ ¥'(ƛ)∪Ǥ(ƛ), & ƛ∈T∩Ƶ. \end{cases}
$$

From the definition, it is obvious that if T=∅, then (¥,T)$+_ε$(Ǥ,Ƶ)=(Ǥ,Ƶ), if Z=∅, then (¥,T)$+_ε$(Ǥ,Ƶ)=(¥,T), if T=Z=∅, then (¥,T)$+_ε$(Ǥ,Ƶ)=$∅_∅$.

Example 2. Let E={e_1,e_2,e_3,e_4}be the parameter set, T=e_1{,e_3} and Z={e_2,e_3,e_4} be subsets of E, U={h_1,h_2, h_3,h_4, h_5} be the universal set, (¥,T) and (Ǥ,Ƶ) be the SSs over U

as(¥,T)={(e_1,{h_2,h_5}),(e_3,{h_1,h_2,h_5})},(₢,ƻ)={(e_2,{h_1,h_4,h_5}),{(e_3,{h_2,h_3,h_4}),(e_4, h_3,h_5})}. Here let (¥,T) $+_\varepsilon$(₢,ƻ)=(ϰ,T∪ƻ), where for all ƛ∈T∪ƻ,

$$
ϰ(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T\text{-}ƻ, \\ ₢(ƛ), & ƛ∈ƻ\text{-}T, \\ ¥'(ƛ)∪₢(ƛ), & ƛ∈T∩ƻ. \end{cases}
$$

Since T∪ƻ={e_1,e_2,e_3,e_4} and T-ƻ={e_1}, ƻ-T={e_2, e_4}, T∩ƻ={e_3}, thus, ϰ(e_1)=¥(e_1)={h_2,h_5}, ϰ(e_2)=₢(e_2)={h_1,h_4,h_5}, ϰ(e_4)=₢(e_4)={h_3,h_5}, ϰ(e_3)=¥'(e_3)∪₢(e_3)={h_3,h_4}∪{h_2, h_3, h_4}={h_2, h_3, h_4}. Thus,

$$
(¥,T)+_\varepsilon(₢,ƻ) = \{(e_1,\{h_2,h_5\}),\ (e_2,\{h_1,h_4,h_5\}),(e_3,\{h_2,h_3,h_4\}),(e_4,\{\ h_3,h_5\})\}.
$$

Remark 1. In the set ST(U), where T is a fixed subset of E, restricted and extended plus operations coincide. That is, (¥,T) $+_\varepsilon$(₢,T)=(¥,T) $+_R$(₢,T).

Theorem 6. (Algebraic properties of the operation)

I. The set $S_E(U)$ and $S_T(U)$ are closed under $+_\varepsilon$.

Proof: it is clear that $+_\varepsilon$ is a binary operation in $S_E(U)$. That is,

$$
\begin{aligned}
&+_\varepsilon: S_E(U) \times S_E(U) \ \rightarrow\ S_E(U)\\
&\quad\big((¥,T),(₢,ƻ)\big) \rightarrow\ (¥,T)+_\varepsilon(₢,ƻ) = (\mathfrak{H},T\cup ƻ),
\end{aligned}
$$

Namely, when (¥,T) and (₢,ƻ) are SS over U, then so (¥,T) $+_\varepsilon$ (₢,ƻ). Similarly, $S_T(U)$ is closed under $+_\varepsilon$. That is,

$$
\begin{aligned}
&+_\varepsilon: S_T(U) \times S_T(U) \ \rightarrow\ S_T(U)\\
&\quad\big((¥,T),(₢,T)\big) \rightarrow (¥,T)+_\varepsilon(₢,T) = (K,T\cup T) = (K,T),
\end{aligned}
$$

Namely, $+_\varepsilon$ is a binary operation in $S_T(U)$.

II. Let (¥,T), (₢,ƻ) and (ϰ,Ȣ) be SSs over U. If T∩ƻ∩Ȣ=∅, then [(¥,T)$+_\varepsilon$ (₢,ƻ)] $+_\varepsilon$ (ϰ,Ȣ) = (¥,T) $+_\varepsilon$ [(₢,ƻ) $+_\varepsilon$ (ϰ,Ȣ)].

Proof: first, consider the LHS. Let (¥,T)$+_\varepsilon$(₢,ƻ)=(S,T∪ƻ), where for all ƛ∈T∪ƻ,

$$
S(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T\text{-}ƻ, \\ ₢(ƛ), & ƛ∈ƻ\text{-}T, \\ ¥'(ƛ)∪₢(ƛ), & ƛ∈T∩ƻ. \end{cases}
$$

Let (S,T∪ƻ) $+_\varepsilon$($\mathfrak{H}$, Ȣ) =(N,(T∪ƻ)∪Ȣ)), where for all ƛ∈(T∪ƻ)∪Ȣ,

$$
N(ƛ)=\begin{cases} S(ƛ), & ƛ∈(T∪ƻ)\text{-}Ȣ, \\ ϰ(ƛ), & ƛ∈Ȣ\text{-}(T∪Ȣ), \\ S'(ƛ)∪ϰ(ƛ), & ƛ∈(T∪ƻ)∩Ȣ. \end{cases}
$$

Thus,

$$
N(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈(T\text{-}ƻ)\text{-}Ȣ=T∩ƻ'∩Ȣ', \\ ₢(ƛ), & ƛ∈(ƻ\text{-}T)\text{-}Ȣ=T'∩ƻ∩Ȣ', \\ ¥'(ƛ)∪₢(ƛ), & ƛ∈(T∩ƻ)\text{-}Ȣ=T∩ƻ∩Ȣ', \\ ϰ(ƛ), & ƛ∈Ȣ\text{-}(T∪ƻ)=T'∩ƻ'∩Ȣ, \\ ¥'(ƛ)∪ϰ(ƛ), & ƛ∈(T\text{-}ƻ)∩Ȣ=T∩ƻ'∩Ȣ, \\ ₢'(ƛ)∪ϰ(ƛ), & ƛ∈(ƻ\text{-}T)∩Ȣ=T'∩ƻ∩Ȣ, \\ [¥(ƛ)∩₢'(ƛ)]∪ϰ(ƛ), & ƛ∈(T∩ƻ)∩Ȣ=T∩ƻ∩Ȣ. \end{cases}
$$

Now consider the RHS. Let (₢,ƻ) $+_\varepsilon$(ϰ,Ȣ)=(R,ƻ∪Ȣ), where for all ƛ∈ƻ∪Ȣ,

$$R(ƛ)=\begin{cases} Ɠ(ƛ), & ƛ∈ƻ-Ɣ,\\ ℋ(ƛ), & ƛ∈Ɣ-ƻ,\\ Ɠ'(ƛ)∪ℋ(ƛ), & ƛ∈ƻ∩Ɣ.\end{cases}$$

Let (¥,T) $+_ε$(R,ƻ∪Ɣ) =(L,(T∪(ƻ∪Ɣ)), where for all ƛ∈T∪ƻ∪Ɣ,

$$L(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-(ƻ∪Ɣ),\\ R(ƛ), & ƛ∈(ƻ∪Ɣ)-T,\\ ¥'(ƛ)∪R(ƛ), & ƛ∈T∩(ƻ∪Ɣ).\end{cases}$$

Hence,

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-(ƻ∪Ɣ)=T∩ƻ'∩Ɣ',\\ Ɠ(ƛ) & ƛ∈(ƻ-Ɣ)-T=T'∩ƻ∩Ɣ',\\ ℋ(ƛ), & ƛ∈(Ɣ-ƻ)-T=T'∩ƻ'∩Ɣ,\\ Ɠ'(ƛ)∪ℋ(ƛ), & ƛ∈(ƻ∩Ɣ)-T=T'∩ƻ∩Ɣ,\\ ¥'(ƛ)∪Ɠ(ƛ), & ƛ∈T∩(ƻ-Ɣ)=T∩ƻ∩Ɣ',\\ ¥'(ƛ)∪ℋ(ƛ), & ƛ∈T∩(Ɣ-ƻ)=T∩ƻ'∩Ɣ,\\ ¥'(ƛ)∪[Ɠ'(ƛ)∪ℋ(ƛ)], & ƛ∈T∩(ƻ∩Ɣ)=T∩ƻ∩Ɣ.\end{cases}$$

It is observed that (N,(T∪ƻ)∪Ɣ)=(L,T∪(ƻ∪Ɣ)), where T∩ƻ∩Ɣ=∅. That is, in $S_E(U)$, $+_ε$ is associative under certain conditions.

III. Let (¥,T), (Ɠ,T) and (ℋ,T) be SSs over U. Then, [(¥,T) $+_ε$ (Ɠ,T)] $+_ε$ (ℋ,T) ≠ (¥,T) $+_ε$ [(Ɠ,T) $+_ε$ (ℋ,T)].

Proof: the proof follows from *Remark 1* and *Theorem 1* (III). That is in $S_T(U)$, where T is a fixed subset of E, $+_ε$ is not associative.

IV. Let (¥,T) and (Ɠ,ƻ) be SSs over U. Then, (¥,T)$+_ε$(Ɠ,ƻ)≠(Ɠ,ƻ) $+_ε$(¥,T).

Proof: let (¥,T) $+_ε$(Ɠ,ƻ)=(ℋ,T∪ƻ), where for all ƛ∈T∪ƻ,

$$ℋ(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-ƻ,\\ Ɠ(ƛ), & ƛ∈ƻ-T,\\ ¥'(ƛ)∪Ɠ(ƛ), & ƛ∈T∩ƻ.\end{cases}$$

Let (Ɠ,ƻ) $+_ε$(¥,T)=(S,ƻ∪T), where for all ƛ∈ƻ∪T,

$$S(ƛ)=\begin{cases} Ɠ(ƛ), & ƛ∈ƻ-T,\\ ¥(ƛ), & ƛ∈T-ƻ,\\ Ɠ'(ƛ)∪¥(ƛ), & ƛ∈ƻ∩T.\end{cases}$$

Thus, (¥,T) $+_ε$ (Ɠ,ƻ)≠(Ɠ,ƻ) $+_ε$(¥,T). If, ƻ∩T=∅, then (¥,T) $+_ε$ (Ɠ,ƻ)=(Ɠ,ƻ)$+_ε$(¥,T). Moreover, it is obvious that (¥,T) $+_ε$ (Ɠ,T)≠(Ɠ,T)$+_ε$(¥,T). That is, in $S_E(U)$ and $S_T(U)$, $+_ε$ is not commutative.

V. Let (¥,T) be a SS over U. Then, (¥,T) $+_ε$(¥,T)= U_T.

Proof: the proof follows from *Remark 1* and *Theorem 1* (V). That is, in $S_E(U)$, $+_ε$ is not idempotent.

VI. Let (¥,T) be a SS over U. Then, (¥,T) $+_ε$ $∅_T$=$(¥,T)^r$.

Proof: the proof follows from *Remark 1* and *Theorem 1* (VI).

VII. Let (¥,T) be a SS over U. Then, $∅_T$$+_ε$ (¥,T)= U_T.

Proof: the proof follows from *Remark 1* and *Theorem 1* (VII).

VIII. Let (¥,T) be a SS over U. Then, (¥,T)$+_ε$ $∅_∅$=(¥,T).

Proof: let $∅_∅$=(S,∅) and (¥,T) $+_ε$(S, ∅)=(ℋ,T∪∅), where for all ƛ∈T∪∅=T,

$$ℋ(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-∅=T,\\ S(ƛ), & ƛ∈∅ -T=∅,\\ ¥'(ƛ)∪S(ƛ), & ƛ∈T∩∅=∅,\end{cases}$$

Thus, for all $\lambda\in T$, $\mathcal{H}(\lambda)=¥(\lambda)$, $(\mathcal{H},T)=(¥,T)$.

IX. Let $(¥,T)$ be a SS over U. Then, $\emptyset_\emptyset +_\varepsilon (¥,T)=(¥,T)$.

Proof: let $\emptyset_\emptyset=(S,\emptyset)$ and $(S,\emptyset)+_\varepsilon(¥,T)=(\mathcal{H},\emptyset\cup T)$, where for all $\lambda\in\emptyset\cup T=T$,

$$\mathcal{H}(\lambda)=\begin{cases} S(\lambda), & \lambda\in\emptyset\text{-}T=\emptyset,\\ ¥(\lambda), & \lambda\in T\text{-}\emptyset=T,\\ S'(\lambda)\cup ¥(\lambda), & \lambda\in\emptyset\cap T=\emptyset,\end{cases}$$

Thus, for all $\lambda\in T$, $\mathcal{H}(\lambda)=¥(\lambda)$, $(\mathcal{H},T)=(¥,T)$.

By *Theorem 6* (VIII) and (9), we can conclude that in $S_E(U)$, the identity element of $+_\varepsilon$ is the SS $\emptyset_\emptyset$. In classical set theory, it is well-known that $A\cup B=\emptyset\Leftrightarrow A=\emptyset$ and $B=\emptyset$. Thus, it is evident that in $S_E(U)$, we can not find $(\mathfrak{C},K)\in S_E(U)$ such that $(¥,T)+_\varepsilon(\mathfrak{C},K)=(\mathfrak{C},K)+_\varepsilon(¥,T)=\emptyset_\emptyset$, as this situation requires that $T\cup K=\emptyset$ and thus, $T=\emptyset$ and $K=\emptyset$. Since in $S_E(U)$, the only SS with an empty parameter set is $\emptyset_\emptyset$, it follows that only the identity element $\emptyset_\emptyset$ has an inverse and its inverse is its own, as usual. Thus, in $S_E(U)$, any other element except $\emptyset_\emptyset$ does not have an inverse for the operation $+_\varepsilon$.

Corollary 1. Let $(¥,T)$, $(\mathfrak{C},\mathcal{Z})$, and $(\mathcal{H},\mathcal{Y})$ be the elements of SE(U). By *Theorem 6* (I), (II), (IV), (VIII), and (IX), $(S_E(U),+_\varepsilon)$ is a non-commutative monoid whose identity is $\emptyset_\emptyset$, where $T\cap\mathcal{Z}\cap\mathcal{Y}=\emptyset$. Since $(S_A(U),+_\varepsilon)$ does not have associative property, where A is a fixed subset of E; this algebraic structure can not be a semigroup.

X. Let $(¥,T)$ be a SS over U. Then, $(¥,T)+_\varepsilon U_T=U_T$.

Proof: the proof follows from *Remark 1* and *Theorem 1* (XIII). That is, U_T is the right absorbing element for $+_\varepsilon$ in $S_T(U)$.

XI. Let $(¥,T)$ be a SS over U. Then, $U_T+_\varepsilon(¥,T)=(¥,T)$.

Proof: the proof follows from *Remark 1* and *Theorem 1* (XIV). That is, U_T is the left identity element for $+_\varepsilon$ in $S_T(U)$.

XII. Let $(¥,T)$ be a SS over U. Then, $(¥,T)+_\varepsilon U_E=U_E$.

Proof: let $U_E=(T,E)$, where for all $\lambda\in E$, $T(\lambda)=U$. Assume that $(¥,T)+_\varepsilon(T,E)=(\mathcal{H},T\cup E)$, where for all $\lambda\in T\cup E=E$,

$$\mathcal{H}(\lambda)=\begin{cases} ¥(\lambda), & \lambda\in T\text{-}E,\\ T(\lambda), & \lambda\in E\text{-}T,\\ ¥'(\lambda)\cup T(\lambda), & \lambda\in T\cap E.\end{cases}$$

Thus,

$$\mathcal{H}(\lambda)=\begin{cases} ¥(\lambda), & \lambda\in T\text{-}E=\emptyset,\\ U, & \lambda\in E\text{-}T=T',\\ U, & \lambda\in T\cap E=T.\end{cases}$$

Hence, for all $\lambda\in E$, $\mathcal{H}(\lambda)=U$, and so $(\mathcal{H},E)=U_E$. That is, U_E is the right absorbing element for $+_\varepsilon$ in $S_E(U)$. Here note that $U_E+_\varepsilon(¥,T)\neq U_E$, that is U_E is not the left absorbing element for $+_\varepsilon$ in $S_E(U)$. Indeed, let $U_E=(T,E)$ and $(T,E)+_\varepsilon(¥,T)=(K,T\cup E)$, where for all $\lambda\in T\cup E=E$,

$$K(\lambda)=\begin{cases} T(\lambda), & \lambda\in E\text{-}T,\\ ¥(\lambda), & \lambda\in T\text{-}E,\\ T'(\lambda)\cup ¥(\lambda), & \lambda\in E\cap T.\end{cases}$$

Thus,

$$K(ƛ)=\begin{cases} U & ƛ\in E\text{-}T= T', \\ ¥(ƛ), & ƛ\in T\text{-}E=\emptyset, \\ ¥(ƛ), & ƛ\in T\cap E=T. \end{cases}$$

Hence, $(K,E) \neq U_E$.

XIII. Let (¥,T) be a SS over U. Then, $(¥,T)+_\varepsilon (¥,T)r=(¥,T)r$.

Proof: the proof follows from *Remark 1* and *Theorem 1* (XIX). That is, every relative complement of the SS is its own right absorbing element for the operation $+_\varepsilon$ in $S_E(U)$.

XIV. Let (¥,T) be a SS over U. Then, $(¥,T)r +_\varepsilon(¥,T)=(¥,T)$.

Proof: the proof follows from *Remark 1* and *Theorem 1* (XX). That is, every relative complement of the SS is its own left identity element for the operation $+_\varepsilon$ in $S_E(U)$.

XV. Let (¥,T) and (Ꞓ,Ꙃ) be SSs over U. Then, $[(¥,T) +_\varepsilon(Ꞓ,Ꙃ)]r=(¥,T) \overset{*}{\underset{\backslash}{\sim}} (Ꞓ,Ꙃ)$.

Proof: let $(¥,T) +_\varepsilon (Ꞓ,Ꙃ)=(ℋ,T\cup Ꙃ)$, where for all $ƛ\in T\cup Ꙃ$,

$$ℋ(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T\text{-}Ꙃ, \\ Ꞓ(ƛ), & ƛ\in Ꙃ\text{-}T, \\ ¥'(ƛ)\cup Ꞓ(ƛ), & ƛ\in T\cap Ꙃ. \end{cases}$$

Let $(ℋ,T\cup Ꙃ)r =(K,T\cup Ꙃ)$, for all $ƛ\in T\cup Ꙃ$,

$$K(ƛ)=\begin{cases} ¥'(ƛ), & ƛ\in T\text{-}Ꙃ, \\ Ꞓ'(ƛ), & ƛ\in Ꙃ\text{-}T, \\ ¥(ƛ)\cap Ꞓ'(ƛ), & ƛ\in T\cap Ꙃ. \end{cases}$$

Thus, $(K,T\cup Ꙃ)= (¥,T) \overset{*}{\underset{\backslash}{\sim}} (Ꞓ,Ꙃ)$.

XVI. Let (¥,T) and (Ꞓ,T) be SSs over U. Then, $(¥,T) +_\varepsilon(Ꞓ, T)= \emptyset_T \Leftrightarrow (¥, T) =U_T$ and $(Ꞓ, T) = \emptyset_T$.

Proof: the proof follows from *Remark 1* and *Theorem 1* (XXII).

XVII. Let (¥,T) and (Ꞓ,Ꙃ) be SSs over U. Then, $\emptyset_T \cong(¥,T)+_\varepsilon (Ꞓ,Ꙃ)$, $\emptyset_Z \cong(¥,T)+_\varepsilon (Ꞓ,Ꙃ)$, $\emptyset_Z \cong (Ꞓ,Ꙃ)+_\varepsilon (¥,T)$, $\emptyset_T \cong (Ꞓ,Ꙃ)+_\varepsilon (¥,T)$. Moreover, $(¥,T) +_\varepsilon (Ꞓ,Ꙃ) \cong U_{T\cup Ꙃ}$ and $(Ꞓ,Ꙃ) +_\varepsilon(¥,T) \cong U_{Ꙃ\cup T}$.

XVIII. Let (¥,T) and (Ꞓ,T) be SSs over U. Then, $(¥,T)r \cong (¥, T)+_\varepsilon (Ꞓ, T)$ and $(Ꞓ,T) \cong(¥,T) +_\varepsilon (Ꞓ,T)$.

Proof: the proof follows from *Remark 1* and Theorem 1 (XXV).

XIX. Let (¥,T), (Ꞓ,T), and (ℋ,T) be SSs over U. If $(¥,T) \cong (Ꞓ, T)$, then $(Ꞓ,T) +_\varepsilon(ℋ,T) \cong(¥,T) +_\varepsilon(ℋ,T)$ and $(ℋ,Ꙃ) +_\varepsilon(¥,T) \cong(ℋ,Ꙃ) +_\varepsilon(Ꞓ,T)$.

Proof: if $(¥,T) \cong (Ꞓ, T)$, then $(Ꞓ,T) +_\varepsilon(ℋ,T) \cong(¥,T) +_\varepsilon (ℋ,T)$ is obvious from *Remark 1* and *Theorem 1* (XXVI) and (XXVII). Let $(¥,T) \cong (Ꞓ, T)$, where for all $ƛ\in T$, $¥(ƛ)\subseteq Ꞓ(ƛ)$. Let $(ℋ,Ꙃ) +_\varepsilon(¥,T)=(Y,Ꙃ\cup T)$, where for all $ƛ\in Ꙃ\cup T$,

$$Y(ƛ)=\begin{cases} ℋ(ƛ), & ƛ\in Ꙃ\text{-}T, \\ ¥(ƛ), & ƛ\in T\text{-}Ꙃ, \\ ℋ'(ƛ)\cup ¥(ƛ), & ƛ\in Ꙃ\cap T. \end{cases}$$

Let $(ℋ,Ꙃ) +_\varepsilon(Ꞓ,T)=(Y,Ꙃ\cup T)$, where for all $ƛ\in Ꙃ\cup T$,

$$W(ƛ)=\begin{cases} ℋ(ƛ), & ƛ\in Ꙃ\text{-}T, \\ Ꞓ(ƛ), & ƛ\in T\text{-}Ꙃ, \\ ℋ'(ƛ)\cup Ꞓ(ƛ), & ƛ\in Ꙃ\cap T. \end{cases}$$

If $ƛ \in Ꙁ$-T, then Y(ƛ)= $\mathfrak{H}$(ƛ) and W(ƛ)=$\mathfrak{H}$(ƛ), thus Y(ƛ)= $\mathfrak{H}$(ƛ) ⊆ $\mathfrak{H}$(ƛ) =W(ƛ). If ƛ∈T-Ꙁ, then Y(ƛ)= ¥(ƛ) and W(ƛ)=Ꞓ(ƛ), thus Y(ƛ)= ¥(ƛ) ⊆ Ꞓ(ƛ)=W(ƛ). If ƛ∈T∩Ꙁ, then Y(ƛ)=$\mathfrak{H}$'(ƛ)∪¥(ƛ) and W(ƛ)=$\mathfrak{H}$'(ƛ)∪Ꞓ(ƛ), thus Y(ƛ)= $\mathfrak{H}$'(α) ∪ F(α) ⊆ $\mathfrak{H}$'(ƛ)∪Ꞓ(ƛ)=W(ƛ). Thus, for all ƛ∈Ꙁ∪T, Y(ƛ)⊆W(ƛ). Hence, ($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$(¥,T) $\tilde{\subseteq}$($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$(Ꞓ,T).

XX. Let (¥,T), (Ꞓ,T) and ($\mathfrak{H}$,Ꙁ) be SSs over U. If ($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$(¥,T) $\tilde{\subseteq}$($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$(Ꞓ,T), then (¥,T) $\tilde{\subseteq}$ (Ꞓ,T) needs not be true. Similarly, if (Ꞓ,T) $+_{\varepsilon}$($\mathfrak{H}$,T) $\tilde{\subseteq}$(¥,T) $+_{\varepsilon}$($\mathfrak{H}$,T), then (¥,T) $\tilde{\subseteq}$ (Ꞓ,T) needs not be true.

Proof: let E={e_1,e_2,e_3,e_4, e_5, e_6} be the parameter set, T={e_1,e_3} and Z={e_1,e_3, e_5} be the subsets of E, U ={h_1,h_2, h_3,h_4, h_5} be the universal set, and (¥,T), (Ꞓ,T) and ($\mathfrak{H}$,Ꙁ), be SSs over U such that

$$(¥,T)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\},(Ꞓ,T)=\{(e_1,\{h_2\}),(e_3,\{h_1,h_2\})\},(\mathfrak{H},Ꙁ)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_2\})\}.$$

let ($\mathfrak{H}$,Ꙁ)$+_{\varepsilon}$(¥,T)=(L,Ꙁ∪T), where for all ƛ ∈ Ꙁ ∪ T={e_1, e_3, e_5}, L(e_1)=$\mathfrak{H}$'(e_1)∪¥(e_1)=U, L(e_3)=$\mathfrak{H}$'(e_3)∪¥(e_3)=U and L(e_5)=H(e_5)={h_2}. Thus, ($\mathfrak{H}$,Ꙁ)$+_{\varepsilon}$(¥,T) ={(e_1,U), (e_3,U), (e_5, {h_2})}. Now let ($\mathfrak{H}$,Ꙁ)$+_{R}$(Ꞓ,T)=(WꙀ ∪ T,), where for all ƛ ∈ Ꙁ ∪ T={e_1, e_3, e_5}, W(e_1)=$\mathfrak{H}$'(e_1)∪Ꞓ(e_1)=U, W(e_3)=$\mathfrak{H}$'(e_3)∪Ꞓ(e_3)=U, and W(e_5)=$\mathfrak{H}$(e_5)= {h_2}. Hence, ($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$ (Ꞓ,T) ={(e_1, U), (e_3,U), (e_5, {h_2})}.

Thus, it is observed that ($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$(¥,T) $\tilde{\subseteq}$($\mathfrak{H}$,Ꙁ) $+_{\varepsilon}$(Ꞓ,T), but (¥,T) is not a soft subset of (Ꞓ,T). Similarly, if (Ꞓ,T)$+_{\varepsilon}$($\mathfrak{H}$,T)$\tilde{\subseteq}$(¥,T) $+_{\varepsilon}$($\mathfrak{H}$,T), then (¥,T) $\tilde{\subseteq}$ (Ꞓ,T) needs not be true can be shown by choosing ($\mathfrak{H}$,T)={(e_1,U),(e_3,U)} in the above example.

XXI. Let (¥,T), (Ꞓ,T), (K,T), and (L,T) be SSs over U. If (¥,T) $\tilde{\subseteq}$ (Ꞓ,T) and (K,T) $\tilde{\subseteq}$ (L,T), then (Ꞓ,T)$+_{\varepsilon}$(K,T) $\tilde{\subseteq}$(¥,T)$+_{\varepsilon}$ (L,T) and (L,T)$+_{\varepsilon}$(¥,T) $\tilde{\subseteq}$(K,T) $+_{\varepsilon}$ (Ꞓ,T).

Proof: the proof follows from *Remark 1* and *Theorem 1* (XVIII).

Theorem 7. Let (¥,T), (Ꞓ,Ꙁ), and ($\mathfrak{H}$,Ꮡ) be SSs over U. Then, extended plus operation distributes over other SS operations as follows:

Theorem 8. Let (¥,T), (Ꞓ,Ꙁ), and ($\mathfrak{H}$,Ꮡ) be SSs over U. Then, extended plus operation distributes over restricted SS operations as follows:

LHS distributions

I. If T∩(ZΔᏘ)=∅, then (¥,T) $+_{\varepsilon}$[(Ꞓ,Ꙁ) $\cup_R$ ($\mathfrak{H}$,Ꮡ)]=[(¥,T) $+_{\varepsilon}$(Ꞓ,Ꙁ)]$\cup_R$[(¥,T) $+_{\varepsilon}$($\mathfrak{H}$,Ꮡ)].

Proof: consider first the LHS. Let (Ꞓ,Ꙁ) $\cup_R$($\mathfrak{H}$,Ꮡ)=(M,Ꙁ∩Ꮡ), where for all ƛ∈Ꙁ∩Ꮡ, M(ƛ)=Ꞓ(ƛ)∪$\mathfrak{H}$(ƛ). Let (¥,T) $+_{\varepsilon}$(M,Ꙁ∩Ꮡ)=(N,T∪(Ꙁ∩Ꮡ)), where for all ƛ∈T∪(Ꙁ∩Ꮡ),

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T\text{-}(Ꙁ\cap Ꮡ),\\ M(ƛ), & ƛ\in (Ꙁ\cap Ꮡ)\text{-}T,\\ ¥'(ƛ)\cup Ꮡ(ƛ), & ƛ\in T\cap(Ꙁ\cap Ꮡ).\end{cases}$$

Thus,

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T\text{-}(Ꙁ\cap Ꮡ),\\ Ꞓ(ƛ)\cup\mathfrak{H}(ƛ), & ƛ\in (Ꙁ\cap Ꮡ)\text{-}T,\\ ¥'(ƛ)\cup [Ꞓ(ƛ)\cup\mathfrak{H}(ƛ)], & ƛ\in T\cap(Ꙁ\cap Ꮡ).\end{cases}$$

Now consider the RHS, i.e. [(¥,T)$+_{\varepsilon}$(Ꞓ,Ꙁ)] $\cup_R$[(¥,T)$+_{\varepsilon}$($\mathfrak{H}$,Ꮡ)]. (¥,T)$+_{\varepsilon}$(Ꞓ,Ꙁ)=(M,T∪Ꙁ), where for all ƛ∈T∪Ꙁ,

$$M(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T\text{-}Ꙁ,\\ Ꞓ(ƛ), & ƛ\in Ꙁ\text{-}T,\\ ¥'(ƛ)\cup Ꞓ(ƛ), & ƛ\in T\cap Ꙁ.\end{cases}$$

Let (¥,T) $+_{\varepsilon}$ ($\mathfrak{H}$,Ꮡ)=(K,T∪Ꮡ), where for all ƛ∈T∪Ꮡ,

$$K(\lambda)=\begin{cases} ¥(\lambda), & \lambda\in T-ℵ,\\ ℋ(\lambda), & \lambda\in ℵ-T,\\ ¥'(\lambda)\cup ℋ(\lambda), & \lambda\in T\cap ℵ.\end{cases}$$

Assume that $(M,T\cup Z)\cup_R(K,T\cup ℵ)=(W,(T\cup Z)\cap(T\cup ℵ))$, where for all $\lambda\in(T\cup Z)\cap(T\cup ℵ)$, $W(\lambda)=T(\lambda)\cap K(\lambda)$. Thus,

$$W(\lambda)=\begin{cases} ¥(\lambda)\cup ¥(\lambda), & \lambda\in(T-Z)\cap(T-ℵ)=T\cap Z'\cap ℵ',\\ ¥(\lambda)\cup ℋ(\lambda), & \lambda\in(T-Z)\cap(ℵ-T)=\emptyset,\\ ¥(\lambda)\cup[¥'(\lambda)\cup ℋ(\lambda)], & \lambda\in(T-Z)\cap(T\cap ℵ)= T\cap Z'\cap ℵ,\\ 𝔊(\lambda)\cup ¥(\lambda), & \lambda\in(Z-T)\cap(T-ℵ)= \emptyset,\\ 𝔊(\lambda)\cup ℋ(\lambda), & \lambda\in(Z-T)\cap(ℵ-T) =T'\cap Z\cap ℵ,\\ 𝔊(\lambda)\cup[¥'(\lambda)\cup ℋ(\lambda)], & \lambda\in(Z-T)\cap(T\cap ℵ)= \emptyset,\\ [¥'(\lambda)\cup 𝔊(\lambda)]\cup F(\lambda), & \lambda\in(T\cap Z)\cap(T-ℵ)= T\cap Z\cap ℵ',\\ [¥'(\lambda)\cup 𝔊(\lambda)]\cup ℋ(\lambda), & \lambda\in(T\cap Z)\cap(ℵ-T)= \emptyset,\\ [¥'(\lambda)\cup 𝔊(\lambda)]\cup[¥'(\lambda)\cup ℋ(\lambda)], & \lambda\in(T\cap Z)\cap(T\cap ℵ)= T\cap Z\cap ℵ.\end{cases}$$

Hence,

$$W(\lambda)=\begin{cases} ¥(\lambda), & \lambda\in T\cap Z'\cap ℵ',\\ U, & \lambda\in T\cap Z'\cap ℵ,\\ 𝔊(\lambda)\cup ℋ(\lambda), & \lambda\in T'\cap Z\cap ℵ,\\ U, & \lambda\in T\cap Z\cap ℵ',\\ ¥'(\lambda)\cup 𝔊(\lambda)\cup ℋ(\lambda), & \lambda\in T\cap Z\cap ℵ.\end{cases}$$

When considering the $T-(Z\cap ℵ)$ in the function N, since $T-(Z\cap ℵ)=T-(Z\cap ℵ)'$, if an element is in the complement of $(Z\cap ℵ)$, then it is either in $Z-ℵ$, or $ℵ-Z$, or $(Z\cup ℵ)'$. Thus, if $\alpha\in T-(Z\cap ℵ)$, then $\alpha\in T\cap Z\cap ℵ'$ or $\alpha\in T\cap Z'\cap ℵ$ or $\alpha\in T\cap Z'\cap ℵ'$. Therefore, N=W under the condition $T\cap Z'\cap ℵ= T\cap Z\cap ℵ'=\emptyset$, that is $T\cap(Z\Delta ℵ)=\emptyset$.

Here, if $Z\cap ℵ=\emptyset$ and $T\cap(Z\Delta ℵ)=\emptyset$. then $N(\lambda)=W(\lambda)=¥(\lambda)$, thus N is equal to W again. Similarly, if $(T\cup Z)\cap(T\cup ℵ)=T\cup(Z\cap ℵ)=\emptyset$, that is $T=\emptyset$ and $Z\cap ℵ=\emptyset$, then $(N,T\cup(Z\cap ℵ))=(W,(T\cup Z)\cap(T\cup ℵ))=\emptyset_\emptyset$. That is, in the theorem, there is no condition that the intersection of the parameter sets of the SSs whose restricted difference will be calculated must be different from empty.

II. If $T\cap(Z\Delta ℵ)=\emptyset$, then $(¥,T) +_\varepsilon[(𝔊,Z) \cap_R (ℋ,ℵ)] = [(¥,T) +_\varepsilon(𝔊,Z)] \cap_R[(¥,T) +_\varepsilon (ℋ,ℵ)]$.

RHS distributions

I. If $T\cap Z\cap ℵ=\emptyset$, then $[(¥,T) \cap_R(𝔊,Z)]+_\varepsilon(ℋ,ℵ)= [(¥,T)+_\varepsilon(ℋ,ℵ)] \cap_R [(𝔊,Z) +_\varepsilon (ℋ,ℵ)]$.

Proof: consider first the LHS. Let $(¥,T)\cap_R(𝔊,Z)=(R,T\cap Z$, where for all $\lambda\in T\cap Z$, $R(\lambda)=¥(\lambda)\cap 𝔊(\lambda)$. Let $(R,T\cap Z)+_\varepsilon(ℋ,ℵ)=(L,(T\cap Z)\cup ℵ)$, where for all $\lambda\in(T\cap Z)\cup ℵ$,

$$L(\lambda)=\begin{cases} R(\lambda), & \lambda\in(T\cap Z)-ℵ,\\ ℋ(\lambda), & \lambda\in ℵ-(T\cap Z),\\ R'(\lambda)\cup ℋ(\lambda), & \lambda\in(T\cap Z)\cap ℵ.\end{cases}$$

Thus,

$$L(\lambda)=\begin{cases} ¥(\lambda)\cap 𝔊(\lambda), & \lambda\in(T\cap Z)-ℵ,\\ ℋ(\lambda), & \lambda\in ℵ-(T\cap Z),\\ [¥'(\lambda)\cup 𝔊'(\lambda)]\cup ℋ(\lambda), & \lambda\in(T\cap Z)\cap ℵ.\end{cases}$$

Now consider the RHS, i.e. $[(¥,T)+_\varepsilon(ℋ,ℵ)]\cap_R[(𝔊,Z) +_\varepsilon (ℋ,ℵ)]$. Let $(¥,T)+_\varepsilon(ℌ,ℵ)=(S,T\cup ℵ)$, where for all $\lambda\in T\cup ℵ$,

$$S(\lambda)=\begin{cases} ¥(\lambda), & \lambda\in T-ℵ,\\ ℋ(\lambda), & \lambda\in ℵ-T,\\ ¥'(\lambda)\cup ℋ(\lambda), & \lambda\in T\cap ℵ.\end{cases}$$

Let (Ǥ,ƻ) $+_\varepsilon$ (ℋ,Ꝩ)=(K,ƻ∪Ꝩ), where for all ƛ∈ƻ∪Ꝩ,

$$K(ƛ)=\begin{cases} Ǥ(ƛ), & ƛ\in ƻ-Ꝩ,\\ ℋ(ƛ), & ƛ\in Ꝩ-ƻ,\\ Ǥ'(ƛ)\cup ℋ(ƛ), & ƛ\in ƻ\cap Ꝩ.\end{cases}$$

Assume that (S,T∪ƻ)$\cap_R$(K,ƻ∪Ꝩ)=(W,(T∪ƻ)∩(ƻ∪Ꝩ)), where for all ƛ∈(T∪ƻ)∩(ƻ∪Ꝩ), where for W(ƛ)=S(ƛ)∩K(ƛ). Thus,

$$W(ƛ)=\begin{cases} ¥(ƛ)\cap Ǥ(ƛ), & ƛ\in(T-Ꝩ)\cap(ƻ-Ꝩ)=T\cap ƻ\cap Ꝩ',\\ ¥(ƛ)\cap ℋ(ƛ), & ƛ\in(T-Ꝩ)\cap(Ꝩ-ƻ)=\emptyset,\\ ¥(ƛ)\cap[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in(T-Ꝩ)\cap(ƻ\cap Ꝩ)=\emptyset,\\ ℋ(ƛ)\cap Ǥ(ƛ), & ƛ\in(Ꝩ-T)\cap(ƻ-Ꝩ)=\emptyset,\\ ℋ(ƛ)\cap ℋ(ƛ), & ƛ\in(Ꝩ-T)\cap(Ꝩ-ƻ)=T'\cap ƻ'\cap Ꝩ,\\ ℋ(ƛ)\cap[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in(Ꝩ-T)\cap(ƻ\cap Ꝩ)=T'\cap ƻ\cap Ꝩ,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cap Ǥ(ƛ), & ƛ\in(T\cap Ꝩ)\cap(ƻ-Ꝩ)=\emptyset,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cap ℋ(ƛ), & ƛ\in(T\cap Ꝩ)\cap(Ꝩ-ƻ)=T\cap ƻ'\cap Ꝩ,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cap[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in(T\cap Ꝩ)\cap(ƻ\cap Ꝩ)=T\cap ƻ\cap Ꝩ.\end{cases}$$

Therefore,

$$W(ƛ)=\begin{cases} ¥(ƛ)\cap Ǥ(ƛ), & ƛ\in T\cap ƻ\cap Ꝩ',\\ ℋ(ƛ), & ƛ\in T'\cap ƻ'\cap Ꝩ,\\ ℋ(ƛ), & ƛ\in T'\cap ƻ\cap Ꝩ,\\ ℋ(ƛ), & ƛ\in T\cap ƻ'\cap Ꝩ,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cap[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in T\cap ƻ\cap Ꝩ.\end{cases}$$

When considering Ꝩ-(T∩ƻ) in the function L, since Ꝩ-(T∩ƻ)=Ꝩ∩(T∩ƻ)', if an element is in the complement of (T∩ƻ), then it is either in T-ƻ, or in ƻ-T or in (T∪ ƻ)'. Thus, if α∈Ꝩ-(T∩ƻ), then either α∈Ꝩ∩T∩ƻ' or α∈ Ꝩ∩ƻ∩T' or α∈ Ꝩ∩T'∩ƻ'. Therefore, L=W under the condition T∩ƻ∩Ꝩ=∅.

Here, if T∩ƻ=∅, then L(ƛ)=W(ƛ)=ℋ(ƛ), thus N is equal to W again. Similarly, if (T∪Ꝩ)∩(ƻ∪Ꝩ)=(T∩ ƻ)∪Ꝩ=∅, that is T∩ƻ=∅ and M=∅, then (L,(T∩ƻ)∪Ꝩ)=(W,(T∪Ꝩ)∩(ƻ∪Ꝩ))=$\emptyset_\emptyset$. That is, in the theorem, there is no condition that the intersection of the parameter sets of the SSs whose restricted difference will be calculated must be different from empty.

II. If T∩ƻ∩Ꝩ=(TΔZ)∩ Ꝩ=∅, then [(¥,T) $\cup_R$(Ǥ,ƻ)]$+_\varepsilon$(ℋ,Ꝩ)=[(¥,T)$+_\varepsilon$(ℋ,Ꝩ)]$\cup_R$[(Ǥ,ƻ) $+_\varepsilon$(ℋ,Ꝩ)].

Corollary 2. (S_E(U),$\cap_R$,$+_\varepsilon$) is an additive idempotent non-commutative (left) near semiring with unity and zero but without zero symmetric properties and under certain conditions.

Proof: Ali et al. [6] showed that (S_E(U),$\cap_R$) is a commutative, idempotent monoid with identity U_E, that is a bounded semilattice (hence a semigroup). By *Corollary 1*, (S_E(U), $+_\varepsilon$) is a non-commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition T∩ƻ∩Ꝩ=∅, where (¥,T), (Ǥ,ƻ) and (ℋ,Ꝩ) are SSs over U. Moreover, $+_\varepsilon$ distributes over $\cap_R$ from LHS under T∩(ZΔꝨ)=∅ and $(F,A)+_\varepsilon U_E = U_E$, that is, U_E is the right absorbing element for the operation $+_\varepsilon$ in S_E(U). Thus, (S_E(U),$\cap_R$,$+_\varepsilon$) is an additive idempotent non-commutative (left) near semiring with unity and zero under the condition T∩ƻ∩Ꝩ=T∩(ZΔꝨ)=∅. Moreover, since U_E+_ε(¥,A)≠ U_E, (S_E(U),$\cap_R$,$+_\varepsilon$) is a (left) near semiring without zero symmetric property and under certain conditions.

Corollary 3. (S_E(U),$\cap_R$,$+_\varepsilon$) is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Proof: Ali et al. [6] showed that (S_E(U),$\cap_R$) is a commutative, idempotent monoid with identity U_E, that is a bounded semilattice (hence a semigroup). By *Corollary 1*, (S_E(U), $+_\varepsilon$) is a non-commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition T∩ƻ∩Ꝩ=∅, where (¥,T), (Ǥ,ƻ) and (ℋ,Ꝩ) are SSs over U. Moreover, $+_\varepsilon$ distributes over $\cap_R$ from LHS under T∩(ZΔꝨ)=∅ and $+_\varepsilon$ distributes over $\cap_R$ form

RHS under the condition T∩₴∩₰=∅. Consequently, under the condition T∩₴∩₰=T∩(ZΔ₰)=∅, (S_E(U),$\cap_R$,$+_\varepsilon$) is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Corollary 4. (S_E(U),$\cup_R$,$+_\varepsilon$) is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Proof: Ali et al. [6] showed that (S_E(U),$\cup_R$) is a commutative, idempotent monoid with identity $\emptyset_\emptyset$, that is a bounded semilattice (hence a semigroup). By *Corollary 1*, (S_E(U), $+_\varepsilon$) is a non-commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition T∩₴∩₰=∅, where (¥,T), (₢,₴) and (ℌ,₰) are SSs over U. Moreover, $+_\varepsilon$ distributes over $\cup_R$ from LHS under (TΔZ)∩₰=∅ and $+_\varepsilon$ distributes over $\cup_R$ form RHS under the condition T∩₴∩₰=(TΔZ)∩₰=∅. Consequently, under the condition T∩₴∩₰=T∩(ZΔ₰)=(TΔZ)∩₰=∅, (S_E(U),$\cup_R$,$+_\varepsilon$) is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Theorem 9. Let (¥,T), (₢,₴), and (ℌ,₰) be SSs over U. Then, extended plus operation distributes over other extended SS operations as follows:

LHS distributions

I. If T∩(ZΔ₰)=∅, then (¥,T)$+_\varepsilon$[(₢,₴) $\cup_\varepsilon$(ℌ,₰)]=[(¥,T)$+_\varepsilon$(₢,₴)]$\cup_\varepsilon$[(¥,T)$+_\varepsilon$(ℌ,₰)].

Proof: first, consider the LHS. Let (₢,₴) $\cup_\varepsilon$(ℌ,₰)=(R,₴∪₰), where for all ƛ∈₴∪₰,

$$M(ƛ)=\begin{cases} ₢(ƛ), & ƛ∈₴-₰, \\ ℌ(ƛ), & ƛ∈₰-₴, \\ ₢(ƛ)∪ℌ(ƛ), & ƛ∈₴∩₰. \end{cases}$$

Let (¥,T)$+_\varepsilon$(R,₴∪₰) =(N,(T∪(₴∪₰)), where for all ƛ∈T∪(₴∪₰),

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-(₴∪₰), \\ M(ƛ), & ƛ∈(₴∪₰)-T, \\ ¥'(ƛ)∪₰(ƛ), & ƛ∈T∩(₴∪₰). \end{cases}$$

Thus,

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-(₴∪₰)=T∩₴'∩₰', \\ ₢(ƛ), & ƛ∈(₴-₰)-T=T'∩₴∩₰', \\ ℌ(ƛ), & ƛ∈(₰-₴)-T=T'∩₴'∩₰, \\ ₢(ƛ)∪ℌ(ƛ), & ƛ∈(₴∩₰)-T=T'∩₴∩₰, \\ ¥'(ƛ)∪₢(ƛ), & ƛ∈T∩(₴-₰)=T∩₴∩₰', \\ ¥'(ƛ)∪ℌ(ƛ), & ƛ∈T∩(₰-₴)=T∩₴'∩₰, \\ ¥'(ƛ)∪[₢(ƛ)∪ℌ(ƛ)], & ƛ∈T∩(₴∩₰)=T∩₴∩₰. \end{cases}$$

Now consider the RHS i.e. [(¥,T)$+_\varepsilon$(₢,₴)]$\cup_\varepsilon$[(¥,T) $+_\varepsilon$ (ℌ,₰)]. Let (¥,T) $+_\varepsilon$ (₢,₴)=(K,T∪₴) where for all ƛ∈T∪₴,

$$K(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-₴, \\ ₢(ƛ), & ƛ∈₴-T, \\ ¥'(ƛ)∪₢(ƛ), & ƛ∈T∩₴. \end{cases}$$

Let (¥,T) $+_\varepsilon$ (ℌ,₰)=(S,T∪₰), where for all ƛ∈T∪₰,

$$S(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-₰, \\ ℌ(ƛ), & ƛ∈₰-T, \\ ¥'(ƛ)∪ℌ(ƛ), & ƛ∈T∩₰. \end{cases}$$

Let (K,T∪₴) $\cup_\varepsilon$(S,T∪₰)=(L,(T∪₴)∪(T∪₰)), where for all ƛ∈(T∪₴)∪(T∪₰),

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ∈(T∪ƶ)-(T∪ℵ),\\ S(ƛ), & ƛ∈(T∪ℵ)-(T∪ƶ),\\ K(ƛ)∪S(ƛ), & ƛ∈(T∪ƶ)∩(T∪ℵ).\end{cases}$$

Thus,

$$L(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈(T-ƶ)-(T∪ℵ)=\emptyset,\\ Ɠ(ƛ), & ƛ∈(ƶ-T)-(T∪ℵ)=T'∩ƶ∩ℵ',\\ ¥'(ƛ)∪Ɠ(ƛ), & ƛ∈(T∩ƶ)-(T∪ℵ)=\emptyset,\\ ¥(ƛ), & ƛ∈(T-ℵ)-(T∪ƶ)=\emptyset,\\ ℋ(ƛ), & ƛ∈(ℵ-T)-(T∪ƶ)=T'∩ƶ'∩ℵ,\\ ¥'(ƛ)∪ℋ(ƛ), & ƛ∈(T∩ℵ)-(T∪ƶ)=\emptyset,\\ ¥(ƛ)∪¥(ƛ), & ƛ∈(T-ƶ)∩(T-ℵ)=T∩ƶ'∩ℵ',\\ ¥(ƛ)∪ℋ(ƛ), & ƛ∈(T-ƶ)∩(ℵ-T)=\emptyset,\\ ¥(ƛ)∪[¥'(ƛ)∪ℋ(ƛ)], & ƛ∈(T-ƶ)∩(T∩ℵ)=T∩ƶ'∩ℵ,\\ Ɠ(ƛ)∪¥(ƛ), & ƛ∈(ƶ-T)∩(T-ℵ)=\emptyset,\\ Ɠ(ƛ)∪ℋ(ƛ), & ƛ∈(ƶ-T)∩(ℵ-T)= T'∩ƶ∩ℵ,\\ Ɠ(ƛ)∪[¥'(ƛ)∪ℋ(ƛ)], & ƛ∈(ƶ-T)∩(T∩ℵ)=\emptyset,\\ [¥'(ƛ)∪Ɠ(ƛ)]∪¥(ƛ), & ƛ∈(T∩ƶ)∩(T-ℵ)=T∩ƶ∩ℵ',\\ [¥'(ƛ)∪Ɠ(ƛ)]∪ℋ(ƛ), & ƛ∈(T∩ƶ)∩(ℵ-T)=\emptyset,\\ [¥'(ƛ)∪Ɠ(ƛ)]∪[¥'(ƛ)∪ℋ(ƛ)], & ƛ∈(T∩ƶ)∩(T∩ℵ)=T∩ƶ∩ℵ.\end{cases}$$

Thus,

$$L(ƛ)=\begin{cases} Ɠ(ƛ), & ƛ∈T'∩ƶ∩ℵ',\\ ℋ(ƛ), & ƛ∈T'∩ƶ'∩ℵ,\\ ¥(ƛ), & ƛ∈T∩ƶ'∩ℵ',\\ U, & ƛ∈T∩ƶ'∩ℵ,\\ Ɠ(ƛ)∪ℋ(ƛ), & ƛ∈T'∩ƶ∩ℵ,\\ U, & ƛ∈T∩ƶ∩ℵ',\\ ¥'(ƛ)∪Ɠ(ƛ)∪ℋ(ƛ), & ƛ∈T∩ƶ∩ℵ.\end{cases}$$

Hence, N=L, where T∩ƶ∩ℵ'=T∩ƶ'∩ℵ=∅. It is obvious that the condition T∩ƶ∩ℵ' =T∩ƶ'∩ℵ=∅ equals the condition T∩(ZΔℵ)=∅.

II. If T∩(ZΔℵ)=∅, then $(¥,T)+_{ε}[(Ɠ,ƶ) ∩_{ε}(ℋ,ℵ)]=[(¥,T)+_{ε}(Ɠ,ƶ)] ∩_{ε}[(¥,T) +_{ε} (ℋ,ℵ)]$.

RHS distributions

I. If T∩ƶ∩ℵ=∅, then $[(¥,T) ∪_{ε} (Ɠ,ƶ)]+_{ε}(ℋ,ℵ)= [(¥,T)+_{ε}(ℋ,ℵ)] ∪_{ε} [(Ɠ,ƶ) +_{ε} (ℋ,ℵ)]$.

Proof: first, consider the LHS. Let $(¥,T) ∪_{ε}(Ɠ,ƶ)=(R,T∪ƶ)$, where for all ƛ∈T∪ƶ,

$$R(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T-ƶ,\\ Ɠ(ƛ), & ƛ∈ƶ-T,\\ ¥(ƛ)∪Ɠ(ƛ), & ƛ∈T∩ƶ.\end{cases}$$

Let $(R,T∪ƶ) +_{ε} (ℋ,ℵ) =(N,(T∪ƶ)∪ℵ)$, where for all ƛ∈(T∪ƶ)∪ℵ,

$$N(ƛ)=\begin{cases} R(ƛ), & ƛ∈(T∪ƶ)-ℵ,\\ ℋ(ƛ), & ƛ∈ℵ-(T∪ƶ),\\ R'(ƛ)∪ℋ(ƛ), & ƛ∈(T∪ƶ)∩ℵ.\end{cases}$$

Thus,

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈(T-ƶ)-ℵ=T∩ƶ'∩ℵ',\\ Ɠ(ƛ), & ƛ∈(ƶ-T)-ℵ=T'∩ƶ∩ℵ',\\ ¥(ƛ)∪Ɠ(ƛ), & ƛ∈(T∩ƶ)-ℵ=T∩ƶ∩ℵ',\\ ℋ(ƛ), & ƛ∈ℵ-(T∪ƶ)=T'∩ƶ'∩ℵ,\\ ¥'(ƛ)∪ℋ(ƛ), & ƛ∈(T-ƶ)∩ℵ=T∩ƶ'∩ℵ,\\ Ɠ'(ƛ)∪ℋ(ƛ), & ƛ∈(ƶ-T)∩ℵ =T'∩ƶ∩ℵ,\\ [¥'(ƛ)∩Ɠ'(ƛ)]∪ℋ(ƛ), & ƛ∈(T∩ƶ)∩ℵ=T∩ƶ∩ℵ.\end{cases}$$

Now consider the RHS, i.e. $[(¥,T)+_{\varepsilon}(ℋ,ℵ)]\cup_{\varepsilon}[(Ǥ,Ƶ)+_{\varepsilon}(ℋ,ℵ)]$. Let $(¥,T)+_{\varepsilon}(ℋ,ℵ)=(K,T\cup Ƶ)$, where for all $ƛ\in T\cup ℵ$,

$$K(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T-ℵ,\\ ℋ(ƛ), & ƛ\in ℵ-T,\\ ¥'(ƛ)\cup ℋ(ƛ), & ƛ\in T\cap ℵ.\end{cases}$$

Let $(Ǥ,Ƶ)+_{\varepsilon}(ℋ,ℵ)=(S,T\cup ℵ)$, where for all $ƛ\in Ƶ\cup ℵ$,

$$S(ƛ)=\begin{cases} Ǥ(ƛ), & ƛ\in Ƶ-ℵ,\\ ℋ(ƛ), & ƛ\in ℵ-Ƶ,\\ Ǥ'(ƛ)\cup ℋ(ƛ), & ƛ\in Ƶ\cap ℵ.\end{cases}$$

Assume that $(K,T\cup ℵ)\cup_{\varepsilon}(S,Ƶ\cup ℵ)=(L,(T\cup ℵ)\cup(Ƶ\cup ℵ))$, where for all $ƛ\in(T\cup ℵ)\cup(Ƶ\cup ℵ)$,

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ\in(T\cup ℵ)-(Ƶ\cup ℵ),\\ S(ƛ), & ƛ\in(Ƶ\cup ℵ)-(T\cup ℵ),\\ K(ƛ)\cup S(ƛ), & ƛ\in(T\cup ℵ)\cap(Ƶ\cup ℵ).\end{cases}$$

Thus,

$$L(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in(T-ℵ)-(Ƶ\cup ℵ)=T\cap Ƶ'\cap ℵ',\\ ℋ(ƛ), & ƛ\in(ℵ-T)-(Ƶ\cup ℵ)=\emptyset,\\ ¥'(ƛ)\cup ℋ(ƛ), & ƛ\in(T\cap ℵ)-(Ƶ\cup ℵ)=\emptyset,\\ Ǥ(ƛ), & ƛ\in(Ƶ-ℵ)-(T\cup ℵ)=T'\cap Ƶ\cap ℵ',\\ ℋ(ƛ), & ƛ\in(ℵ-Ƶ)-(T\cup ℵ)=\emptyset,\\ Ǥ'(ƛ)\cup ℋ(ƛ), & ƛ\in(Ƶ\cap ℵ)-(T\cup ℵ)=\emptyset,\\ ¥(ƛ)\cup Ǥ(ƛ), & ƛ\in(T-ℵ)\cap(Ƶ-ℵ)=T\cap Ƶ\cap ℵ',\\ ¥(ƛ)\cup ℋ(ƛ), & ƛ\in(T-ℵ)\cap(ℵ-Ƶ)=\emptyset,\\ ¥(ƛ)\cup[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in(T-ℵ)\cap(Ƶ\cap ℵ)=\emptyset,\\ ℋ(ƛ)\cup Ǥ(ƛ), & ƛ\in(ℵ-T)\cap(Ƶ-ℵ)=\emptyset,\\ ℋ(ƛ)\cup ℋ(ƛ), & ƛ\in(ℵ-T)\cap(ℵ-Ƶ)=T'\cap Ƶ'\cap ℵ,\\ ℋ(ƛ)\cup[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in(ℵ-T)\cap(Ƶ\cap ℵ)=T'\cap Ƶ\cap ℵ,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cup Ǥ(ƛ), & ƛ\in(T\cap ℵ)\cap(Ƶ-ℵ)=\emptyset,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cup ℋ(ƛ), & ƛ\in(T\cap ℵ)\cap(ℵ-Ƶ)=T\cap Ƶ'\cap ℵ,\\ [¥'(ƛ)\cup ℋ(ƛ)]\cup[Ǥ'(ƛ)\cup ℋ(ƛ)], & ƛ\in(T\cap ℵ)\cap(Ƶ\cap ℵ)=T\cap Ƶ\cap ℵ.\end{cases}$$

Hence,

$$L(ƛ)=\begin{cases} ¥(ƛ) & ƛ\in T\cap Ƶ'\cap ℵ'\\ Ǥ(ƛ), & ƛ\in T'\cap Ƶ\cap ℵ',\\ ¥(ƛ)\cup Ǥ(ƛ), & ƛ\in T\cap Ƶ\cap ℵ',\\ ℋ(ƛ), & ƛ\in T'\cap Ƶ'\cap ℵ,\\ Ǥ'(ƛ)\cup ℋ(ƛ), & ƛ\in T'\cap Ƶ\cap ℵ,\\ ¥'(ƛ)\cup ℋ(ƛ), & ƛ\in T\cap Ƶ'\cap ℵ,\\ [¥'(ƛ)\cup Ǥ'(ƛ)]\cup ℋ(ƛ), & ƛ\in T\cap Ƶ\cap ℵ.\end{cases}$$

Therefore, N=L, where $T\cap Ƶ\cap ℵ=\emptyset$.

II. If $(T\triangle Z)\cap ℵ=T\cap Ƶ\cap ℵ=\emptyset$, then $[(¥,T)\cap_{\varepsilon}(Ǥ,Ƶ)]+_{\varepsilon}(ℋ,ℵ)=[(¥,T)+_{\varepsilon}(ℋ,ℵ)]\cap_{\varepsilon}[(Ǥ,Ƶ)+_{\varepsilon}(ℋ,ℵ)]$.

Corollary 5. $(S_E(U),\cup_{\varepsilon},+_{\varepsilon})$ is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cup_{\varepsilon})$ is a commutative, idempotent monoid with identity $\emptyset_{\emptyset}$, that is a bounded semilattice (hence a semigroup). By *Corollary 1*, $(S_E(U),+_{\varepsilon})$ is a non-commutative monoid (hence a semigroup) whose identity is $\emptyset_{\emptyset}$ under the condition $T\cap Ƶ\cap ℵ=\emptyset$, where $(¥,T)$, $(Ǥ,Ƶ)$ and $(ℋ,ℵ)$ are SSs over U. Moreover, $+_{\varepsilon}$ distributes over $\cup_{\varepsilon}$ from LHS under $T\cap(Z\Delta ℵ)=\emptyset$, and $+_{\varepsilon}$ distributes over $\cup_{\varepsilon}$ form RHS under the condition $T\cap Ƶ\cap ℵ=\emptyset$. Consequently, under the condition $T\cap Ƶ\cap ℵ=T\cap(Z\Delta ℵ)=\emptyset$, $(S_E(U),\cup_{\varepsilon},+_{\varepsilon})$ is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Corollary 6. $(S_E(U),\cap_\varepsilon,+_\varepsilon)$ is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Proof: Ali et al. [6] showed that $(S_E(U),\cap_\varepsilon)$ is a commutative, idempotent monoid with identity $\emptyset_\emptyset$, that is a bounded semilattice (hence a semigroup). By *Corollary 1*, $(S_E(U),+_\varepsilon)$ is a non-commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition $T\cap Ƶ\cap ℵ=\emptyset$, where (¥,T), (Ɠ,Ƶ) and (ℋ,ℵ) are SSs over U. Moreover, $+_\varepsilon$ distributes over $\cap_\varepsilon$ from LHS under $T\cap(Z\Delta ℵ)=\emptyset$, anD $+_\varepsilon$ distributes over $\cap_\varepsilon$ from RHS under the condition $(T\triangle Z)\cap ℵ=T\cap Ƶ\cap ℵ=\emptyset$. Consequently, under the condition $T\cap Ƶ\cap ℵ=T\cap(Z\Delta ℵ)=(T\triangle Z)\cap ℵ=\emptyset$, $(S_E(U),\cap_\varepsilon,+_\varepsilon)$ is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Theorem 10. Let (¥,T), (Ɠ,Ƶ), and (ℋ,ℵ) be SSs over U. Then, the extended plus operation distributes over soft binary piecewise operations as follows:

LHS distribution

I. If $T\cap(Z\Delta ℵ)=\emptyset$, then $(¥,T)+_\varepsilon[(Ɠ,Ƶ)\tilde{\cap}(ℋ,ℵ)] = [(¥,T)+_\varepsilon(Ɠ,Ƶ)]\tilde{\cap}[(¥,T)+_\varepsilon(ℋ,ℵ)]$.

Proof: first, consider the LHS. Let $(Ɠ,Ƶ)\tilde{\cap}(ℋ,ℵ)=(R,Ƶ)$, where for all $ƛ\in Ƶ$,

$$R(ƛ)=\begin{cases} Ɠ(ƛ), & ƛ\in Ƶ-ℵ,\\ Ɠ(ƛ)\cap ℋ(ƛ), & ƛ\in Ƶ\cap ℵ.\end{cases}$$

Let $(¥,T)+_\varepsilon(R,Ƶ)=(N,T\cup Ƶ)$, where for all $ƛ\in T\cup Ƶ$,

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T-Ƶ,\\ R(ƛ), & ƛ\in Ƶ-T,\\ ¥'(ƛ)\cup R(ƛ), & ƛ\in T\cap Ƶ.\end{cases}$$

Thus,

$$N(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T-Ƶ\\ Ɠ(ƛ), & ƛ\in(Ƶ-ℵ)-T=T'\cap Ƶ\cap ℵ',\\ Ɠ(ƛ)\cap ℋ(ƛ), & ƛ\in(Ƶ\cap ℵ)-T=T'\cap Ƶ\cap ℵ,\\ ¥'(ƛ)\cup Ɠ(ƛ), & ƛ\in T\cap(Ƶ-ℵ)=T\cap Ƶ\cap ℵ',\\ ¥'(ƛ)\cup[Ɠ(ƛ)\cap ℋ(ƛ)], & ƛ\in T\cap(Ƶ\cap ℵ)=T\cap Ƶ\cap ℵ.\end{cases}$$

Now consider the RHS, i.e, $[(¥,T+_\varepsilon)\ (Ɠ,Ƶ\tilde{\cap}\ [)]\ (¥,T)+_\varepsilon(ℋ,ℵ)]$. Let $(¥,T)+_\varepsilon(Ɠ,Ƶ)=(K,T\cup Ƶ)$, where for all $ƛ\in T\cup Ƶ$,

$$K(ƛ)=\begin{cases} ¥(ƛ), & ƛ\in T-Ƶ,\\ Ɠ(ƛ), & ƛ\in Ƶ-T,\\ ¥'(ƛ)\cup Ɠ(ƛ), & ƛ\in T\cap Ƶ.\end{cases}$$

Let $(¥,T)+_\varepsilon(ℋ,ℵ)=(S,T\cup ℵ)$, where for all $ƛ\in T\cup ℵ$,

$$S(ƛ)=\begin{cases} ¥(ƛ) & ƛ\in T-ℵ\\ ℋ(ƛ) & ƛ\in ℵ-T\\ ¥'(ƛ)\cup ℋ(ƛ) & ƛ\in T\cap ℵ.\end{cases}$$

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ\in(T\cup Ƶ)-(T\cup ℵ),\\ K(ƛ)\cap S(ƛ), & ƛ\in(T\cup Ƶ)\cap(T\cup ℵ).\end{cases}$$

Let $(K,T\cup Ƶ)\tilde{\cap}(S,T\cup ℵ)=(L,(T\cup Ƶ)\cup(T\cup ℵ))$, where for all $ƛ\in(T\cup Ƶ)\cup(T\cup ℵ)$,

Thus,

$$\text{L}(\text{ƛ})=\begin{cases}\text{¥}(\text{ƛ}), & \text{ƛ}\in(\text{T}-\text{Ƶ})-(\text{T}\cup\text{Ȣ})=\emptyset\\ \text{Ꞓ}(\text{ƛ}), & \text{ƛ}\in(\text{Ƶ}-\text{T})-(\text{T}\cup\text{Ȣ})=\text{T}'\cap\text{Ƶ}\cap\text{Ȣ}',\\ \text{¥}'(\text{ƛ})\cup\text{Ꞓ}(\text{ƛ}), & \text{ƛ}\in(\text{T}\cap\text{Ƶ})-(\text{T}\cup\text{Ȣ})=\emptyset,\\ \text{¥}(\text{ƛ})\cap\text{¥}(\text{ƛ}), & \text{ƛ}\in(\text{T}-\text{Ƶ})\cap(\text{T}-\text{Ȣ})=\text{T}\cap\text{Ƶ}'\cap\text{Ȣ}',\\ \text{¥}(\text{ƛ})\cap\text{ℋ}(\text{ƛ}), & \text{ƛ}\in(\text{T}-\text{Ƶ})\cap(\text{Ȣ}-\text{T})=\emptyset,\\ \text{¥}(\text{ƛ})\cap[\text{¥}'(\text{ƛ})\cup\text{ℋ}(\text{ƛ})], & \text{ƛ}\in(\text{T}-\text{Ƶ})\cap(\text{T}\cap\text{Ȣ})=\text{T}\cap\text{Ƶ}'\cap\text{Ȣ},\\ \text{Ꞓ}(\text{ƛ})\cap\text{¥}(\text{ƛ}), & \text{ƛ}\in(\text{Ƶ}-\text{T})\cap(\text{T}-\text{Ȣ})=\emptyset,\\ \text{Ꞓ}(\text{ƛ})\cap\text{ℋ}(\text{ƛ}), & \text{ƛ}\in(\text{Ƶ}-\text{T})\cap(\text{Ȣ}-\text{T})=\text{T}'\cap\text{Ƶ}\cap\text{Ȣ},\\ \text{Ꞓ}(\text{ƛ})\cap[\text{¥}'(\text{ƛ})\cup\text{ℋ}(\text{ƛ})], & \text{ƛ}\in(\text{Ƶ}-\text{T})\cap(\text{T}\cap\text{Ȣ})=\emptyset,\\ [\text{¥}'(\text{ƛ})\cup\text{Ꞓ}(\text{ƛ})]\cap\text{¥}(\text{ƛ}), & \text{ƛ}\in(\text{T}\cap\text{Ƶ})\cap(\text{T}-\text{Ȣ})=\text{T}\cap\text{Ƶ}\cap\text{Ȣ}',\\ [\text{¥}'(\text{ƛ})\cup\text{Ꞓ}(\text{ƛ})]\cap\text{ℋ}(\text{ƛ}), & \text{ƛ}\in(\text{T}\cap\text{Ƶ})\cap(\text{Ȣ}-\text{T})=\emptyset,\\ [\text{¥}'(\text{ƛ})\cup\text{Ꞓ}(\text{ƛ})]\cap[\text{¥}'(\text{ƛ})\cup\text{ℋ}(\text{ƛ})], & \text{ƛ}\in(\text{T}\cap\text{Ƶ})\cap(\text{T}\cap\text{Ȣ})=\text{T}\cap\text{Ƶ}\cap\text{Ȣ}.\end{cases}$$

Thus,

$$\text{L}(\text{ƛ})=\begin{cases}\text{Ꞓ}(\text{ƛ}), & \text{ƛ}\in\text{T}'\cap\text{Ƶ}\cap\text{Ȣ}',\\ \text{¥}(\text{ƛ}), & \text{ƛ}\in\text{T}\cap\text{Ƶ}'\cap\text{Ȣ}',\\ \text{¥}(\text{ƛ})\cap\text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{T}\cap\text{Ƶ}'\cap\text{Ȣ},\\ \text{Ꞓ}(\text{ƛ})\cap\text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{T}'\cap\text{Ƶ}\cap\text{Ȣ},\\ \text{Ꞓ}(\text{ƛ})\cap\text{¥}(\text{ƛ}), & \text{ƛ}\in\text{T}\cap\text{Ƶ}\cap\text{Ȣ}',\\ \text{¥}'(\text{ƛ})\cup[\text{Ꞓ}(\text{ƛ})\cap\text{ℋ}(\alpha], & \text{ƛ}\in\text{T}\cap\text{Ƶ}\cap\text{Ȣ}.\end{cases}$$

When considering $\text{T}-\text{Ƶ}$ in the function N, since $\text{T}-\text{Ƶ}=\text{T}\cap\text{Ƶ}'$, if an element is in the complement of Z, it is either in $\text{Ȣ}-\text{Ƶ}$, or $(\text{Ȣ}\cup\text{Ƶ})'$. Thus, if $\alpha\in\text{T}-\text{Ƶ}$, then either $\alpha\in\text{T}\cap\text{Ȣ}\cap\text{Ƶ}'$ or $\alpha\in\text{T}\cap\text{Ȣ}'\cap\text{Ƶ}$, hence N=L where $\text{T}\cap\text{Ƶ}\cap\text{Ȣ}'=\text{T}\cap\text{Ƶ}'\cap\text{Ȣ}=\emptyset$. It is obvious that the condition $\text{T}\cap\text{Ƶ}\cap\text{Ȣ}'=\text{T}\cap\text{Ƶ}'\cap\text{Ȣ}=\emptyset$ equals the condition $\text{T}\cap(\text{Z}\Delta\text{Ȣ})=\emptyset$

II. If $\text{T}\cap(\text{Z}\Delta\text{Ȣ})=\emptyset$, then $(\text{¥},\text{T})+_{\varepsilon}[(\text{Ꞓ},\text{Ƶ})\,\tilde{\cup}\,\text{H},\text{Ȣ})]=[(\text{¥},\text{T})+_{\varepsilon}(\text{Ꞓ},\text{Ƶ})]\,\tilde{\cup}\,[(\text{¥},\text{Ȣ})+_{\varepsilon}(\text{Ꞓ},\text{Ƶ})]$.

RHS distributions

I. If $\text{T}'\cap\text{Ƶ}\cap\text{Ȣ}=\text{T}\cap\text{Ƶ}\cap\text{Ȣ}=\emptyset$, then $[(\text{¥},\text{T})\,\tilde{\cup}\,(\text{Ꞓ},\text{Ƶ})]+_{\varepsilon}(\text{ℋ},\text{Ȣ})=[(\text{¥},\text{T})+_{\varepsilon}(\text{ℋ},\text{Ȣ})]\,\tilde{\cup}\,[(\text{Ꞓ},\text{Ƶ})+_{\varepsilon}(\text{ℋ},\text{Ȣ})]$.

Proof: first, consider the LHS of the equality. Let $(\text{¥},\text{T})\,\tilde{\cup}\,(\text{Ꞓ},\text{Ƶ})=(\text{R},\text{T})$, where for all $\text{ƛ}\in\text{T}$,

$$\text{R}(\text{ƛ})=\begin{cases}\text{¥}(\text{ƛ}), & \text{ƛ}\in\text{T}-\text{Ƶ},\\ \text{¥}(\text{ƛ})\cup\text{Ꞓ}(\text{ƛ}), & \text{ƛ}\in\text{T}\cap\text{Ƶ}.\end{cases}$$

Let $(\text{R},\text{T})+_{\varepsilon}(\text{ℋ},\text{Ȣ})=(\text{N},\text{T}\cup\text{Ȣ})$, where for all $\text{ƛ}\in\text{T}\cup\text{Ȣ}$,

$$\text{N}(\text{ƛ})=\begin{cases}\text{R}(\text{ƛ}), & \text{ƛ}\in\text{T}-\text{Ȣ},\\ \text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{Ȣ}-\text{T},\\ \text{R}'(\text{ƛ})\cup\text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{T}\cap\text{Ȣ}.\end{cases}$$

Thus,

$$\text{N}(\text{ƛ})=\begin{cases}\text{¥}(\text{ƛ}), & \text{ƛ}\in(\text{T}-\text{Ƶ})-\text{Ȣ}=\text{T}\cap\text{Ƶ}'\cap\text{Ȣ}',\\ \text{¥}(\text{ƛ})\cup\text{Ꞓ}(\text{ƛ}), & \text{ƛ}\in(\text{T}\cap\text{Ƶ})-\text{Ȣ}=\text{T}\cap\text{Ƶ}\cap\text{Ȣ}',\\ \text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{Ȣ}-\text{T},\\ \text{¥}'(\text{ƛ})\cup\text{ℋ}(\text{ƛ}), & \text{ƛ}\in(\text{T}-\text{Ƶ})\cap\text{Ȣ}=\text{T}\cap\text{Ƶ}'\cap\text{Ȣ},\\ [\text{¥}'(\text{ƛ})\cap\text{Ꞓ}'(\text{ƛ})]\cup\text{ℋ}(\text{ƛ}), & \text{ƛ}\in(\text{T}\cap\text{Ƶ})\cap\text{Ȣ}=\text{T}\cap\text{Ƶ}\cap.\end{cases}$$

Now consider the RHS, i.e. $[(\text{¥},\text{T})+_{\varepsilon}(\text{ℋ},\text{Ȣ})]\,\tilde{\cup}\,[(\text{Ꞓ},\text{Ƶ})+_{\varepsilon}(\text{ℋ},\text{Ȣ})$. Let $(\text{¥},\text{T})+_{\varepsilon}(\text{ℋ},\text{Ȣ})=(\text{K},\text{T}\cup\text{Ȣ})$, where for all $\text{ƛ}\in\text{T}\cup\text{Ȣ}$,

$$\text{K}(\text{ƛ})=\begin{cases}\text{¥}(\text{ƛ}), & \text{ƛ}\in\text{T}-\text{Ȣ},\\ \text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{Ȣ}-\text{T},\\ \text{¥}'(\text{ƛ})\cup\text{ℋ}(\text{ƛ}), & \text{ƛ}\in\text{T}\cap\text{Ȣ}.\end{cases}$$

Let (₲,ƻ) $+_\varepsilon$ (ℋ,Ȣ)=(S,T∪Ȣ), where for all ƛ∈ƻ∪Ȣ,

$$S(ƛ)=\begin{cases} ₲(ƛ), & ƛ∈ƻ\text{-}Ȣ,\\ ℋ(ƛ), & ƛ∈Ȣ\text{-}ƻ,\\ ₲'(ƛ)∪ℋ(ƛ), & ƛ∈ƻ∩Ȣ. \end{cases}$$

Let (K,T∪Ȣ) $\tilde{\cup}$ (S,ƻ∪Ȣ)=(L,(T∪Ȣ)∪(ƻ∪Ȣ)), where for all ƛ∈(T∪Ȣ)∪(ƻ∪Ȣ),

$$L(ƛ)=\begin{cases} K(ƛ), & ƛ∈(T∪Ȣ)\text{-}(ƻ∪Ȣ),\\ K(ƛ)∪S(ƛ), & ƛ∈(T∪Ȣ)∩(ƻ∪Ȣ). \end{cases}$$

Thus,

$$L(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈(T\text{-}Ȣ)\text{-}(ƻ∪Ȣ)=T∩ƻ'∩Ȣ',\\ ℋ(ƛ), & ƛ∈(Ȣ\text{-}T)\text{-}(ƻ∪Ȣ)=∅,\\ ¥'(ƛ)∪ℋ(ƛ), & ƛ∈(T∩Ȣ)\text{-}(ƻ∪Ȣ)=∅,\\ ¥(ƛ)∪₲(ƛ), & ƛ∈(T\text{-}Ȣ)∩(ƻ\text{-}Ȣ)=T∩ƻ∩Ȣ',\\ ¥(ƛ)∪ℋ(ƛ), & ƛ∈(T\text{-}Ȣ)∩(Ȣ\text{-}ƻ)=∅,\\ ¥(ƛ)∪ [₲'(ƛ)∪ℋ(ƛ)], & ƛ∈(T\text{-}Ȣ)∩(ƻ∩Ȣ)=∅,\\ ℋ(ƛ)∪₲(ƛ), & ƛ∈(Ȣ\text{-}T)∩(ƻ\text{-}Ȣ)=∅,\\ ℋ(ƛ)∪ℋ(ƛ), & ƛ∈(Ȣ\text{-}T)∩(Ȣ\text{-}ƻ)= T'∩ƻ'∩Ȣ,\\ ℋ(ƛ)∪ [₲'(ƛ)∪ℋ(ƛ)], & ƛ∈(Ȣ\text{-}T)∩(ƻ∩Ȣ)=T'∩ƻ∩Ȣ,\\ [¥'(ƛ)∪ℋ(ƛ)]∪₲(ƛ), & ƛ∈(T∩Ȣ)∩(ƻ\text{-}Ȣ)=∅,\\ [¥'(ƛ)∪ℋ(ƛ)]∪ℋ(ƛ), & ƛ∈(T∩Ȣ)∩(Ȣ\text{-}ƻ)=T∩ƻ'∩Ȣ,\\ [¥'(ƛ)∪ℋ(ƛ)]∪ [₲'(ƛ)∪ℋ(ƛ)], & ƛ∈(T∩Ȣ)∩(ƻ∩Ȣ)=T∩ƻ∩Ȣ. \end{cases}$$

Hence,

$$L(ƛ)=\begin{cases} ¥(ƛ), & ƛ∈T∩ƻ'∩Ȣ',\\ ¥(ƛ)∪₲(ƛ), & ƛ∈T∩ƻ∩Ȣ',\\ ℋ(ƛ), & ƛ∈T'∩ƻ'∩Ȣ,\\ ₲'(ƛ)∪ℋ(ƛ), & ƛ∈T'∩ƻ∩Ȣ,\\ ¥'(ƛ)∪ℋ(ƛ), & ƛ∈T∩ƻ'∩Ȣ,\\ [¥'(ƛ)∪₲'(ƛ)]∪ℋ(ƛ), & ƛ∈T∩ƻ∩Ȣ. \end{cases}$$

When considering Ȣ-T in the function N, since Ȣ-T=Ȣ∩T', if an element is in the complement of T, then it is either in ƻ-T or (ƻ∪T)'. Thus if α∈Ȣ-T, then α∈ Ȣ∩ƻ∩T' or α∈Ȣ∩ƻ'∩T'. Thus, N=L under T'∩ƻ∩Ȣ= T∩ƻ∩Ȣ =∅.

II. If T∩ƻ'∩Ȣ=T∩ƻ∩Ȣ =∅, then [(¥,T) $\tilde{\cap}$ (₲,ƻ)] $+_\varepsilon$ (ℋ,Ȣ)=[(¥,T)$+_\varepsilon$(ℋ,Ȣ)] $\tilde{\cap}$ [(₲,ƻ) $+_\varepsilon$(ℋ,Ȣ)].

Corollary 7. (S_E(U), $\tilde{\cup}$,$+_\varepsilon$) is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Proof: Yavuz [28] showed that (S_E(U), $\tilde{\cup}$) is an idempotent, non-commutative semigroup (that is, a band) under the condition T∩ƻ'∩Ȣ =∅, where (¥,T), (₲,ƻ) and (ℋ,Ȣ) are SSs.

By *Corollary 1*, (S_E(U), $+_\varepsilon$) is a non-commutative monoid (hence a semigroup) whose identity is $∅_∅$ under the condition T∩ƻ∩Ȣ=∅, where (¥,T), (₲,ƻ) and (ℋ,Ȣ) are SSs over U. Moreover,, $+_\varepsilon$ distributes over $\tilde{\cup}$ from LHS under T∩(ZΔȢ)=∅, and $+_\varepsilon$ distributes over $\tilde{\cup}$ from RHS under the condition T'∩ƻ∩Ȣ=T∩ƻ∩Ȣ=∅. Consequently, under the condition T∩ƻ∩Ȣ=T∩(ZΔȢ)=(TΔZ)∩Ȣ=∅, (S_E(U),$\tilde{\cup}$,$+_\varepsilon$) is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Corollary 8. $(S_E(U), \tilde{\cap}, +_\varepsilon)$ is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

Proof: Yavuz [28] showed that $(S_E(U), \tilde{\cap})$ is an idempotent, non-commutative semigroup (that is, a band) under the condition T∩Ƶ'∩ℵ =∅, where (¥,T), (Ᵹ,Ƶ) and (ℋ,ℵ) are SSs.

By *Corollary 1*, $(S_E(U), +_\varepsilon)$ is a non-commutative monoid (hence a semigroup) whose identity is $\emptyset_\emptyset$ under the condition T∩Ƶ∩ℵ=∅, where (¥,T), (Ᵹ,Ƶ) and (ℋ,ℵ) are SSs over U. Moreover, $+_\varepsilon$ distributes over $\tilde{\cap}$ from LHS under T∩(ZΔℵ)=∅, and $+_\varepsilon$ distributes over $\tilde{\cap}$ from RHS under the condition T'∩Ƶ∩ℵ=T∩Ƶ∩ℵ=∅. Consequently, under the condition T∩Ƶ∩ℵ= T∩Ƶ'∩ℵ= T∩(ZΔℵ)=∅, $(S_E(U), \tilde{\cap}, +_\varepsilon)$ is an additive idempotent non-commutative semiring without zero but with unity under certain conditions.

4 | Conclusion

When working with uncertain objects, parametric tools such as SSs and soft operations are highly effective. Novel insights into the solution of parametric data issues come from the proposal of novel soft operations and the derivation of their algebraic properties and implementations. This study introduces a new restricted and extended SS operation. We aim to contribute to the literature on SS theory by proposing restricted and extended plus operations of SSs and systematically investigating algebraic structures connected with these new SS operations and other SS operations. Remarkably, the algebraic properties of these new soft-set operations are thoroughly examined.

Parametric techniques like SSs and operations are helpful when dealing with uncertain objects. Proposing new soft operations and deriving their algebraic features and implementations provide new insights into solving parametric data problems. In this sense, a novel kind of restricted and extended SS operation, which we call restricted and extended plus operation of SSs, is presented in this work, and by systematically examining the algebraic structures associated with these and other novel SS operations, we contribute o the theory of SS. Specifically, these novel SS operations' algebraic properties are analyzed in detail. A thorough examination of the algebraic structures that the collection of SSs over a universe constructs with these operations is given, considering the algebraic properties of these SS operations and distribution laws. We show that $(S_E(U), +_\varepsilon)$ is a non-commutative monoid with identity $\emptyset_\emptyset$. Additionally, we show that the collection of SSs over the universe, together with extended plus operation and another type of SS operation, form many important algebraic structures such as semiring near semiring:

- *$(S_E(U), \cap_R, +_\varepsilon)$, $(S_E(U), \cup_R, +_\varepsilon)$, $(S_E(U), \cup_\varepsilon, +_\varepsilon)$, $(S_E(U), \cap_\varepsilon, +_\varepsilon)$, $(S_E(U), \tilde{\cap}', +_\varepsilon)$, $(S_E(U), \tilde{\cup}', +_\varepsilon)$ are all additive idempotent non-commutative semiring without zero but with unity under certain conditions.*
- *$(S_E(U), \cap_R, +_\varepsilon)$ is also additive commutative, idempotent, (left) near semirings with zero unity but without zero symmetric property under certain conditions.*

We understand their usefulness by examining the algebraic structures of SSs and new SS operations. This not only presents novel examples for algebraic structures but also has the potential to improve both the classical algebraic literature and the field of SS theory. Further types of new restricted and extended SS operations, corresponding distributions, and characteristics may be investigated in future studies to contribute to the literature.

Author Contributions

Aslıhan Sezgin conceptualized the study, developed the theoretical framework, and wrote the original draft. Fitnat Nur Aybek contributed to the analysis of algebraic structures and the review of existing soft set

operations. Nurcan Bilgili Güngör was responsible for refining the manuscript and provided feedback on the mathematical modeling and applications discussed.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data Availability

No datasets were generated or analyzed during the current study, as it is theoretical in nature. Any inquiries regarding the theoretical framework and methods can be directed to the authors.

Conflicts of Interest

The authors declare no conflicts of interest in relation to this research.

References

[1] Molodtsov, D. (1999). Soft set theory—first results. *Computers & mathematics with applications*, *37*(4-5), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

[2] Maji, P. K., Biswas, R., & Roy, A. R. (2003). Soft set theory. *Computers and mathematics with applications*, *45*(4–5), 555–562. DOI:10.1016/S0898-1221(03)00016-6

[3] Pei, D., & Miao, D. (2005). From soft sets to information systems. *2005 IEEE international conference on granular computing* (Vol. 2005, pp. 617–621). IEEE. DOI: 10.1109/GRC.2005.1547365

[4] Ali, M. I., Feng, F., Liu, X., Min, W. K., & Shabir, M. (2009). On some new operations in soft set theory. *Computers & mathematics with applications*, *57*(9), 1547–1553.

[5] Sezgin, A., & Yavuz, E. (2023). A new soft set operation: soft binary piecewise symmetric difference operation. *Necmettin erbakan üniversitesi fen ve mühendislik bilimleri dergisi*, *5*(2), 189–208.

[6] Ali, M. I., Shabir, M., & Naz, M. (2011). Algebraic structures of soft sets associated with new operations. *Computers and mathematics with applications*, *61*(9), 2647–2654. DOI:10.1016/j.camwa.2011.03.011

[7] Yang, C. F. (2008). A note on "soft set theory"[comput. math. appl. 45 (4--5)(2003) 555--562]. *Computers & mathematics with applications*, *56*(7), 1899–1900.

[8] Wu, F. (2005). A new approach to the theory of turbulence. *Journal of shanghai university*, *9*(3), 189–193. DOI:10.1007/s11741-005-0075-1

[9] Singh, D., & Onyeozili, I. A. (2012). Notes on soft matrices operations. *ARPN journal of science and technology*, *2*(9), 861–869. http://www.ejournalofscience.org

[10] Singh, D., & A. Onyeozili, I. (2012). On some new properties of soft set operations. *International journal of computer applications*, *59*(4), 39–44. DOI:10.5120/9538-3975

[11] Professor D. Singh, P. D. S. (2012). Some results on distributive and absorption properties of soft operations. *IOSR journal of mathematics*, *4*(2), 18–30. DOI:10.9790/5728-0421830

[12] Singh, D., & Onyeozili, I. A. (2012). Some conceptual misunderstandings of the fundamentals of soft set theory 1 1. *ARPN journal of systems and software*, *2*(9), 251–254.

[13] Zhu, P., & Wen, Q. (2013). Operations on soft sets revisited. *Journal of applied mathematics*, *2013*(1), 105752. DOI:10.1155/2013/105752

[14] Sen, J. (2014). On algebraic structure of soft sets. *Annals of fuzzy mathematics and informatics*, *7*(6), 1013–1020.

[15] Saeed, M., Ahsan, M., Siddique, M. K., & Ahmad, M. R. (2020). A study of the fundamentals of hypersoft set theory. *International journal of scientific and technology research*, *3*(4), 132–143.

[16] Husain, S., & Shivani, K. (2018). A study of properties of soft set and its applications. *International research journal of engineering and technology (IRJET)*, *5*(01), 56–2395.

[17] Eren, Ö. F., & Çalışıcı, H. O. some operations of soft sets. (2019). *On some operations of soft sets* [presentation]. The fourth international conference on computational mathematics and engineering sciences (CMES-2019), antalya.

[18] Sezgin, A. & Çağman, N. (2024). A new soft set operation: Complementary soft binary piecewise difference operation. *Osmaniye korkut ata university journal of the institute of science and technology, 7*(1), 58-94. https://doi.org/10.47495/okufbed.1308379

[19] Stojanović, N. S. (2021). A new operation on soft sets: extended symmetric difference of soft sets. *Vojnotehnički glasnik/military technical courier*, *69*(4), 779–791.

[20] Sezgin, A., & Sarıalioğlu, M. (2024). A new soft set operation: complementary soft binary piecewise theta (**θ**) operation. *Journal of kadirli faculty of applied sciences*, *4*(2), 325–357.

[21] Sezgin, A., Çağman, N., Atagün, A. O., & Aybek, F. N. (2023). Complemental binary operations of sets and their application to group theory. *Matrix science mathematic*, *7*(2), 114–121. DOI:10.26480/msmk.02.2023.114.121

[22] Çağman, N. (2021). Conditional complements of sets and their application to group theory. *Journal of new results in science*, *10*(3), 67–74. DOI:10.54187/jnrs.1003890

[23] Aybek, F. (2024). New restricted and extended soft set operations. *MSc, Amasya university, Amasya,Türkiye*.

[24] Akbulut, E. (2024). New type of extended operations of soft set: complementary extended difference and lambda operation. *MSc, Amasya university, Amasya,Türkiye*.

[25] Demirci, A. M. (2024). New type of extended operations of soft set: complementary extended union, plus and theta operation. *MSc, Amasya university, Amasya,Türkiye*.

[26] Sarıalioğlu, M. (2024). *New type of extended operations of soft set: complementary extended intersection, gamma and star operation, MSc, Amasya university, Amasya,Türkiye*.

[27] Çağman, N., & Enginoğlu, S. (2010). Soft set theory and uni--int decision making. *European journal of operational research*, *207*(2), 848–855.

[28] Yavuz, E. (2024). Soft binary piecewise operations and their properties. *MSc, Amasya university, Amasya,Türkiye*.

[29] Sezgin, A., Atagün, A. O., Çağman, N., & Demir, H. (2022). On near-rings with soft union ideals and applications. *New mathematics and natural computation*, *18*(2), 495–511. DOI:10.1142/S1793005722500247

[30] Tunçay, M. & Sezgin, A. (2016). Soft union ring and its applications to ring theory. *International journal of computer applications*, *151*(9), 7-13. 10.5120/ijca2016911867

[31] Sezer, A. S., Çagman, N., & Atagün, A. O. (2015). Uni-soft substructures of groups. *Annals of fuzzy mathematics and informatics*, *9*(2), 235–246.

[32] Sezer, A. S. (2014). Certain characterizations of LA-semigroups by soft sets. *Journal of intelligent and fuzzy systems*, *27*(2), 1035–1046. DOI:10.3233/IFS-131064

[33] Mahmood, T., Rehman, Z. U., & Sezgin, A. (2018). Lattice ordered soft near rings. *Korean journal of mathematics*, *26*(3), 503–517.

[34] Mustuoglu, E., Sezgin, A., & Kaya, Z. (2016). Some characterizations on soft uni-groups and normal soft uni-groups. *International journal of computer applications*, *155*(10), 1–8. DOI:10.5120/ijca2016912412

[35] Özlü, Ş., & Sezgin, A. (2020). Soft covered ideals in semigroups. *Acta universitatis sapientiae, mathematica*, *12*(2), 317–346. DOI:10.2478/ausm-2020-0023

[36] Sezgin, A. (2016). A new approach to semigroup theory I: soft union semigroups, ideals and bi-ideals. *Algebra letters*, *2016*(3), 1–46.

[37] Sezgin, A., & Atagün, A. O. (2016). A new kind of vector space: soft vector space. *Southeast Asian bulletin of mathematic*s, 40(5), 753–770. https://www.researchgate.net/publication/308938468

[38] Jana, C., Pal, M., Karaaslan, F., & Sezgin, A. (2019). (α, β)-soft intersectional rings and ideals with their applications. *New mathematics and natural computation*, *15*(2), 333–350. DOI:10.1142/S1793005719500182

[39] Sezer, A. S. (2014). A new approach to LA-semigroup theory via the soft sets. *Journal of intelligent & fuzzy systems*, *26*(5), 2483–2495.

[40] Atagün A. O., & Sezgin, A. (2018). Soft subnear-rings, soft ideals and soft N-subgroups of near-rings, *Mathematical sciences letters*, *7*(1), 37-42. DOI:10.18576/msl/070106

[41] Sezer, A. S., Çağman, N., & Atagün, A. O. (2014). Soft intersection interior ideals, quasi-ideals and generalized bi-ideals; a new approach to semigroup theory II. *Journal of multiple-valued logic and soft computing*, *23*(1–2), 161–207.

[42] Sezgin, A. (2018). A new view on AG-groupoid theory via soft sets for uncertainty modeling. *Filomat*, *32*(8), 2995–3030. DOI:10.2298/FIL1808995S

[43] Sezgin, A., Çağman, N., & Atagün, A. O. (2017). A completely new view to soft intersection rings via soft uni-int product. *Applied soft computing journal*, *54*, 366–392. DOI:10.1016/j.asoc.2016.10.004

[44] Atagün, A. O., Kamacı, H., Taştekin, İ., & Sezgin, A. (2019). P-properties in near-rings. *Journal of mathematical and fundamental sciences*, *51*(2), 152-167. https://dx.doi.org/10.5614/j.math.fund.sci.2019.51.2.5

[45] Khan, A., Izhar, M., & Sezgin, A. (2017). Characterizations of abel grassmann's groupoids by the properties of their double-framed soft ideals. *International journal of analysis and applications*, *15*(1), 62–74. http://www.etamaths.com

[46] Gulistan, M., Feng, F., Khan, M., & Sezgin, A. (2018). Characterizations of right weakly regular semigroups in terms of generalized cubic soft sets. *Mathematics*, *6*(12), 293. DOI:10.3390/math6120293

[47] Manikantan, T., Ramasamy, P., & Sezgin, A. (2023). Soft Quasi-ideals of soft near-rings. *Sigma*, *41*(3), 565–574.

[48] Atagün, A. O., & Sezgin, A. (2017). Int-soft substructures of groups and semirings with applications. *Applied mathematics and information sciences*, *11*(1), 105–113. DOI:10.18576/amis/110113

[49] Atagün, A. O., & Sezer, A. S. (2015). Soft sets, soft semimodules and soft substructures of semimodules. *Mathematical sciences letters*, *4*(3), 235–242.

[50] Sezer, A. S., Atagün, A. O., & Çağman, N. (2014). N-group SI-action and its applications to N-Group Theory. *Fasciculi mathematici*, *52*, 139-153. https://www.researchgate.net/profile/Aslihan-Sezgin2/publication/263651539_N-group_SI-action_and_its_application_to_Ngroup_theory/links/54353a080cf2bf1f1f283279/N-group-SI-action-and-its-application-to-N-grouptheory.pdf

[51] Riaz, M., Hashmi, M. R., Karaaslan, F., Sezgin, A., Ali Al Shamiri, M. M., & Khalaf, M. M. (2023). Emerging trends in social networking systems and generation gap with neutrosophic crisp soft mapping. *CMES - computer modeling in engineering and sciences*, *136*(2), 1759–1783. DOI:10.32604/cmes.2023.023327

[52] Atagün, A. O., & Sezgin, A. (2018). A new view to near-ring theory: soft near-rings. *Southeast asian journal of mathematics & mathematical sciences*, 14(3). https://openurl.ebsco.com/EPDB%3Agcd%3A6%3A21060918/detailv2?sid=ebsco%3Aplink%3Ascholar&id=ebsco%3Agcd%3A133122391&crl=c

[53] Sezer, A. S., Atagün, A. O., & Çağman, N. (2013). A new view to N-group theory: soft N-groups, *Fasciculi mathematici*, *51*, 123-140. https://www.researchgate.net/profile/Aslihan-Sezgin2/publication/263651532_A_new_view_to_N-group_theory-Soft_Ngroups/links/0046353b68f17da045000000/A-new-view-to-N-group-theory-Soft-N-groups.pdf

[54] Atagün, A. O., & Sezgin, A. (2022). More on prime, maximal and principal soft ideals of soft rings. *New mathematics and natural computation*, *18*(1), 195-207. https://doi.org/10.1142/S1793005722500119

[55] Clifford, A. H. (1954). Bands of semigroups. *Proceedings of the american mathematical society*, *5*(3), 499–504. DOI:10.2307/2031968

[56] Vandiver, H. S. (1936). Note on an associative distributive algebra in which the commutative law of addition does not hold. *Bulletin of the american mathematical society*, *42*(12), 857–859. DOI:10.1090/S0002-9904-1936-06444-X

[57] van Hoorn, W. G., & van Oozelaa, B. (1967). Fundamental notions in the theory of seminearrings. *Compositio mathematica*, *18*(1–2), 65–78.

[58] Pant, S., Dagtoros, K., Kholil, M. I., & Vivas, A. (2024). Matrices: peculiar determinant property. *Optimum science journal*, (1), 1–7. https://www.optimumscience.org/index.php/pub/article/view/9